Ex 1 :

a) $I = \iint_{D_R} 1 \, dx\,dy$

$$= \int_{r=0}^{R} \int_{\theta=-\pi}^{\pi} r \, d\theta \, dr$$

$$= \int_{\theta=-\pi}^{\pi} \int_{r=0}^{R} r \, dr \, d\theta$$

$$= \int_{\theta=-\pi}^{\pi} \frac{R^2}{2} \, d\theta$$

$$= \frac{R^2}{2} \int_{\theta=-\pi}^{\pi} 1 \, d\theta = \frac{R^2}{\cancel{2}} \times \cancel{2}\pi$$

$$= \pi R^2.$$

b) $I = \iint_{D_R} e^{-(x^2+y^2)} \, dx \, dy$

$x = r\cos(\theta) \qquad y = r\sin\theta$

$x^2+y^2 = r^2\cos^2(\theta) + r^2\sin^2(\theta)$

$= r^2 \underbrace{\left(\cos^2(\theta) + \sin^2(\theta)\right)}_{=1} = r^2$

$= \int_{\theta=-\pi}^{\pi} \int_{r=0}^{R} e^{-r^2} \, r \, dr \, d\theta$

$= \int_{\theta=-\pi}^{\pi} \left[-\frac{e^{-r^2}}{2} \right]_{r=0}^{R} d\theta$

$= \int_{\theta=-\pi}^{\pi} \frac{1}{2} - \frac{e^{-R^2}}{2} \, d\theta = \frac{1-e^{-R^2}}{2} \int_{\theta=-\pi}^{\pi} d\theta = \frac{1-e^{-R^2}}{\cancel{2}} \cancel{2}\theta$

$= \theta\left(1-e^{-R^2}\right)$

c) $I = \iiint_{B_R} 1 \, dx\,dy\,dz$

$= 4\pi \int_0^R r^2 \, dr$

$= \frac{4\pi R^3}{3}$.

d) $I = \iiint_{B_R} \frac{1}{\sqrt{x^2+y^2+z^2}} \, dx\,dy\,dz$

$= 4\pi \int_0^R \frac{1}{r} \times r^2 \, dr$

$= 4\pi \int_0^R r \, dr = \frac{4\pi R^2}{2} = 2\pi R^2$

Ex 2 :

a) $f(x,y) = x^3 + 3xy + 4y^2$

$$\nabla f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x^2 + 3y \\ 8y + 3x \end{pmatrix}$$

$$\Delta f(x,y) = 6x + 8$$

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b) $f(x,y,z) = x^2y + y^2z + z^2x$

$$\nabla f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xy + z^2 \\ x^2 + 2yz \\ y^2 + 2xz \end{pmatrix}$$

$$\Delta f(x,y) = 2y + 2z + 2x = 2(x+y+z)$$

c) $\varphi(x,y) = \ln(x^2+y^2)$

$$= g(r) = \ln(r^2) = 2\ln(r)$$

$$\nabla\varphi = \begin{pmatrix} \dfrac{2x}{x^2+y^2} \\ \\ \dfrac{2y}{x^2+y^2} \end{pmatrix} \qquad \begin{pmatrix} g(r) = 2\ln(r) \\ \\ g'(r) = \dfrac{2}{r} \\ \\ g''(r) = -\dfrac{2}{r^2} \end{pmatrix}$$

$$\Delta\varphi = -\frac{2}{r^2} + \frac{1}{r} \times \frac{2}{r}$$

$$= -\frac{2}{r^2} + \frac{2}{r^2} = 0$$

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d) $\varphi(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}} = g(r) = \dfrac{1}{r}$

$$= \left(x^2+y^2+z^2\right)^{-1/2}$$

$$\nabla \varphi = -\frac{1}{2}\left(x^2+y^2+z^2\right)^{-3/2}\begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix}$$

$$= -\left(x^2+y^2+z^2\right)^{-3/2}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \left(\frac{-x}{\sqrt{x^2+y^2+z^2}^{\,3}} ; \frac{-y}{\sqrt{x^2+y^2+z^2}^{\,3}} ; \frac{-z}{\sqrt{x^2+y^2+z^2}^{\,3}}\right)$$

$$g(r) = \frac{1}{r} \qquad g'(r) = -\frac{1}{r^2} \qquad g''(r) = 2r^{-3}$$

$$= \frac{2}{r^3}$$

$$\Delta\varphi = \frac{2}{r^3} + \frac{2}{r} \times \left(-\frac{1}{r^2}\right)$$

$$= \frac{2}{r^3} - \frac{2}{r^3} = 0.$$

Ex 3

$$u(t,x,y) = \sin(2x)\sin(3y)e^{-\lambda t}$$

$$\partial_t u = -\lambda \sin(2x)\sin(3y)e^{-\lambda t}$$

$$\nabla u = \begin{pmatrix} 2\cos(2x)\sin(3y)e^{-\lambda t} \\ 3\sin(2x)\cos(3y)e^{-\lambda t} \end{pmatrix}$$

$$\Delta u = -4\sin(2x)\sin(3y)e^{-\lambda t}$$

$$-9\sin(2x)\sin(3y)e^{-\lambda t}$$

$$= -13\sin(2x)\sin(3y)e^{-\lambda t}$$

Donc λ doit valoir $\underline{13}$

Ex 4:

$$u(t,x,y,z) = e^{i\omega t} e^{-r^2}$$

$$i\partial_t u = -\omega e^{i\omega t} e^{-r^2}$$

$$g(r) = e^{-r^2}$$

$$g'(r) = -2r e^{-r^2}$$

$$g''(r) = -2 e^{-r^2} + 4r^2 e^{-r^2} = e^{-r^2}(4r^2-2)$$

$$\Delta u = e^{i\omega t} e^{-r^2}(4r^2-2-2)$$

$$= 4 e^{i\omega t}(r^2-1)$$

$$i\partial_t u + \Delta u - 4r^2 u$$

$$= e^{i\omega t} e^{-r^2}\left(-\omega + 4(\cancel{r^2}-1) - \cancel{4r^2}\right)$$

$$= e^{i\omega t} e^{-r^2} \underbrace{(-\omega - 4)}_{\text{on veut } 0!}$$

donc $\omega = -4$