Ex 1

$$f(x,y) = x^2 + 2xy - 6x - 2y^2 + 6y - 2$$

$$\nabla f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \partial_x f(x,y) \\ \partial_y f(x,y) \end{pmatrix} = \begin{pmatrix} 2x + 2y - 6 \\ -4y + 2x + 6 \end{pmatrix}$$

$$H_f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \partial_{xx} f(x,y) & \partial_{xy} (x,y) \\ \partial_{yx} f(x,y) & \partial_{yy} (x,y) \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 2 \\ 2 & -4 \end{pmatrix}$$

$$\nabla f \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} = \begin{matrix} 0 \\ 0 \end{matrix} \iff \begin{cases} 2x + 2y - 6 = 0 \\ 2x - 4y + 6 = 0 \end{cases}$$

$$\iff \begin{cases} x + y = 3 \\ x - 2y = -3 \end{cases} \qquad (L_1) - (L_2) : 3y = 6 \iff \boxed{y = 2}$$

L_1 : $x = 3 - y = 3 - 2 = 1$

Donc $(x, y) = (1, 2)$ est un pt critique.

Taylor au point critique $(1,2)$:

$$f(1+h, 2+k) = f(1,2) + \overbrace{\nabla f\begin{pmatrix}1\\2\end{pmatrix}}^{=0}\begin{pmatrix}h\\k\end{pmatrix}$$
$$+ \begin{pmatrix}h & k\end{pmatrix} H_f\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix} + o(h^2+k^2)$$
$$= f(1,2) + \begin{pmatrix}h & k\end{pmatrix} H_f\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}h\\k\end{pmatrix} + o(h^2+k^2)$$

$$\det\left(H_f\begin{pmatrix}1\\2\end{pmatrix}\right) = \det\begin{pmatrix}2 & 2\\ 2 & -4\end{pmatrix} = (2\times -4) - (2\times 2)$$
$$= -8 - 4$$
$$= -12 < 0$$

Donc le point critique $(1,2)$ est un point selle.

Ex 2:

$$PV = nRT \quad (*)$$

$$\Updownarrow$$

$$\underbrace{PV - nRT}_{=: F(P,V,T)} = 0$$

a) $P = \dfrac{nRT}{V} = f(V,T)$

$V = \dfrac{nRT}{P} = g(P,T)$

$T = \dfrac{PV}{nR} = h(P,T)$

b) $$\frac{\partial P}{\partial V} \times \frac{\partial V}{\partial P} = \frac{\partial f}{\partial V} \times \frac{\partial g}{\partial P}$$

$$= \left(-\frac{nRT}{V^2}\right) \times \left(-\frac{nRT}{P^2}\right)$$

$$= + \frac{(nRT)^2}{P^2 V^2} = \frac{(nRT)^2}{(PV)^2} \overset{(*)}{=} \frac{(nRT)^2}{(nRT)^2} = 1.$$

$$\frac{\partial P}{\partial V} \times \frac{\partial V}{\partial T} \times \frac{\partial T}{\partial P}$$

$$= \frac{\partial f}{\partial V} \times \frac{\partial g}{\partial T} \times \frac{\partial h}{\partial P}$$

$$= \left(-\frac{nRT}{V^2}\right)\left(\frac{nR}{P}\right)\left(\frac{V}{nR}\right)$$

$$= \left(-\frac{\cancel{nR}T}{V^{\cancel{2}}}\right)\left(\frac{nR}{P}\right)\left(\frac{\cancel{V}}{\cancel{nR}}\right)$$

$$= -\frac{nRT}{PV} \overset{*}{=} -\frac{nRT}{nRT} = -1.$$

<u>Ex 3</u> : $f(x,y,z) = ax + by + cz + d = 0$

$$x = \frac{-by - cz - d}{a} \qquad z = \frac{-ax - by - d}{c}$$

$$y = \frac{-ax - cz - d}{b}$$

$$\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial x} = \left(-\frac{b}{a}\right)\left(-\frac{a}{b}\right) = 1$$

$$\frac{\partial x}{\partial z} \times \frac{\partial z}{\partial x} = \left(-\frac{c}{a}\right)\left(-\frac{a}{c}\right) = 1$$

$$\frac{\partial y}{\partial z} \times \frac{\partial z}{\partial y} = \left(-\frac{c}{b}\right)\left(-\frac{b}{c}\right) = 1$$

$$\frac{\partial x}{\partial y} \times \frac{\partial y}{\partial z} \times \frac{\partial z}{\partial x} = \left(-\frac{b}{a}\right)\left(-\frac{c}{b}\right)\left(-\frac{a}{c}\right)$$

$$= \left(+\frac{\cancel{b}}{\cancel{a}}\right)\left(+\frac{\cancel{c}}{\cancel{b}}\right)\left(-\frac{\cancel{a}}{\cancel{c}}\right)$$

$$= -1.$$

Ex 4 : $(x_1, x_2) = (x, y) = X$

a) $f : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) \mapsto (xy \,;\, xy^2) \end{cases}$

$$J_f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y & x \\ y^2 & 2xy \end{pmatrix}$$

$$df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y\,dx + x\,dy \\ y^2\,dx + 2xy\,dy \end{pmatrix}$$

$$J_f\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} y & x \\ y^2 & 2xy \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix}$$

$$= \begin{pmatrix} yh + xk \\ y^2h + 2xyk \end{pmatrix}$$

$$df\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} y\, dx\begin{pmatrix} h \\ k \end{pmatrix} + x\, dy\begin{pmatrix} h \\ k \end{pmatrix} \\ y^2\, dx\begin{pmatrix} h \\ k \end{pmatrix} + 2xy\, dy\begin{pmatrix} h \\ k \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} yh + xk \\ y^2 h + 2xyk \end{pmatrix}$$

$$= J_f\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} h \\ k \end{pmatrix}$$

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$$f : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^3 \\ (x,y) \mapsto (x+y \;;\; xy \;;\; x^2+y^2) \end{cases}$$

$$J_f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ y & x \\ 2x & 2y \end{pmatrix}$$

$$df\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} dx + dy \\ y\,dx + x\,dy \\ 2x\,dx + 2y\,dy \end{pmatrix}$$

$$Jf\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ y & x \\ 2x & 2y \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} h + k \\ yh + xk \\ 2xh + 2yk \end{pmatrix}$$

$$df\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} dx\begin{pmatrix} h \\ k \end{pmatrix} + dy\begin{pmatrix} h \\ k \end{pmatrix} \\ y\,dx\begin{pmatrix} h \\ k \end{pmatrix} + x\,dy\begin{pmatrix} h \\ k \end{pmatrix} \\ 2x\,dx\begin{pmatrix} h \\ k \end{pmatrix} + 2y\,dy\begin{pmatrix} h \\ k \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} h + k \\ yh + xk \\ 2xh + 2yk \end{pmatrix} = Jf\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} h \\ k \end{pmatrix}$$

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