The field-road diffusion model: fundamental solution and asymptotic behavior

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(joint work with Matthieu Alfaro and Romain Ducasse)

University of Rouen, Normandy, France

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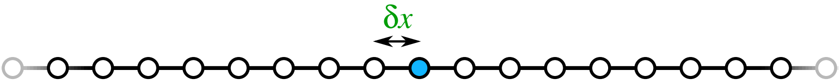
The field-road diffusion model

Let ● be a single individual living along a discrete line



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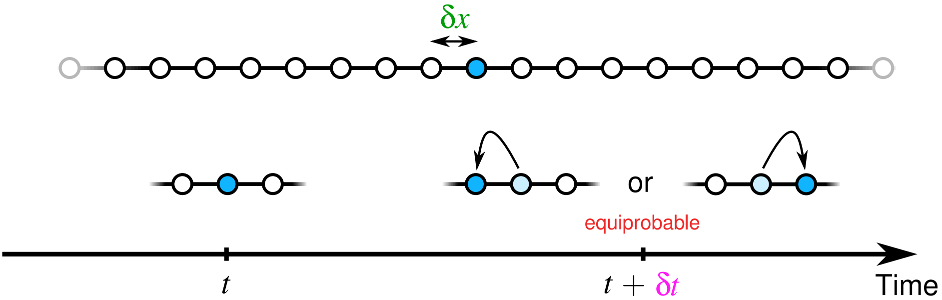

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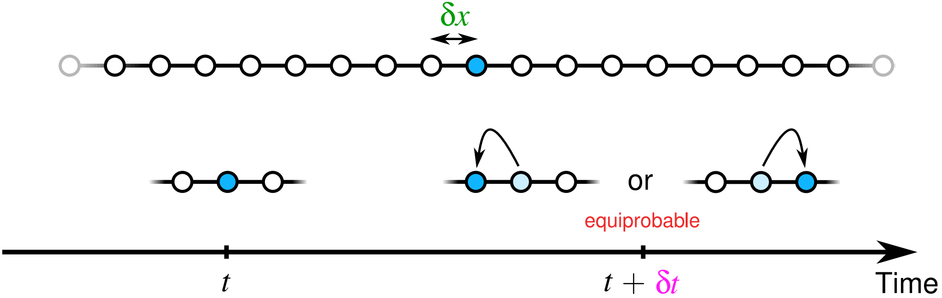

→ independent from the past

Let $v(t,x) = \mathbb{P}($● is in x at time $t)$

$$\frac{v(t+\delta t,x)-v(t,x)}{\delta t} = \frac{\delta x^2}{2\delta t}\,\frac{v(t,x-\delta x)-2v(t,x)+v(t,x+\delta x)}{\delta x^2}$$

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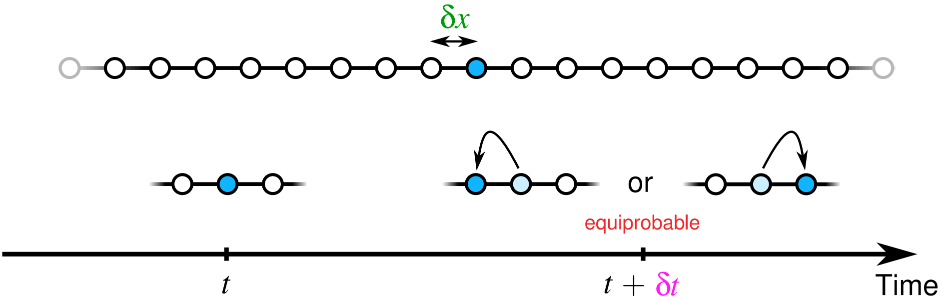

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Say this ratio is constant (equal d), we let $\delta x, \delta t \to 0$.

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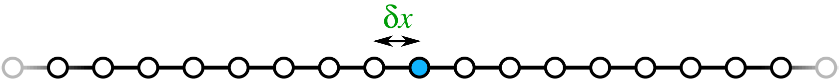
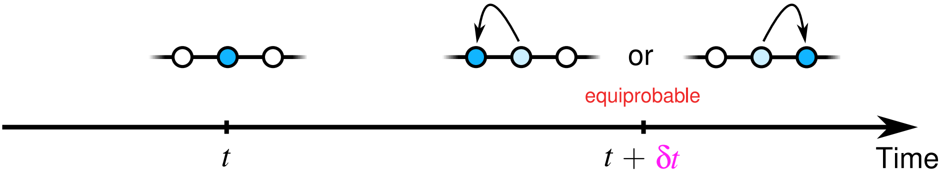
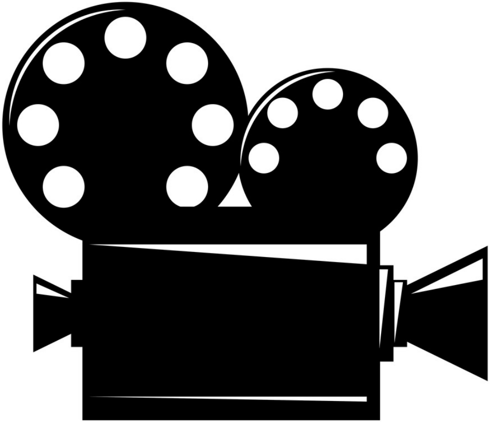




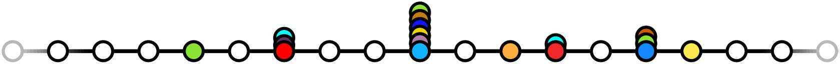
→ independent from the past

Let $v(t,x) = \mathbb{P}($● is in x at time $t)$

$$\underbrace{\partial_t v(t,x) = \boxed{d}\ \partial_{xx} v(t,x)}_{\text{Heat equation}}$$



Take now a population of $n \in \mathbb{N}^*$ individuals



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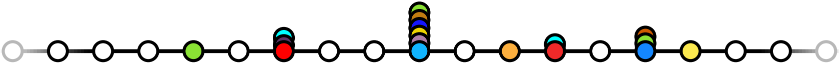

Law of large numbers guaranties that

$$v(t,x) = \text{population density in } x \text{ at time } t$$

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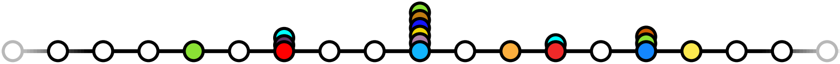
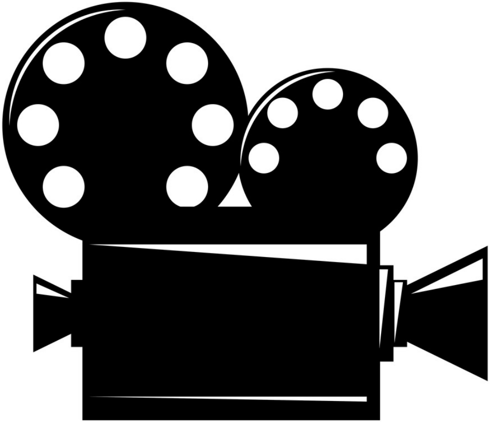


Law of large numbers guaranties that

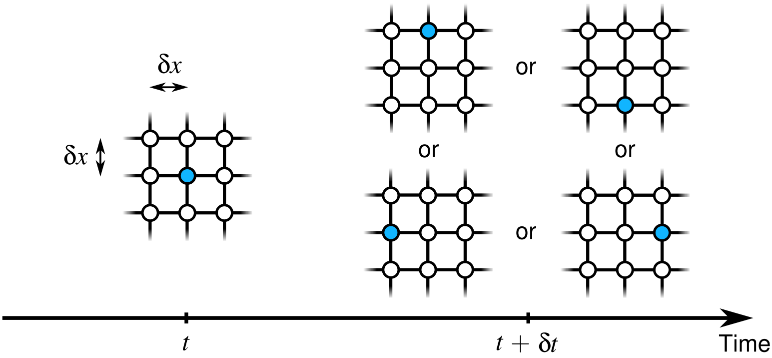
$v(t,x) =$ population density in x at time t satisfies

$$\begin{cases} \partial_t v = d\partial_{xx} v & t > 0, \quad x \in \mathbb{R} \\ v|_{t=0} = v_0 & \quad x \in \mathbb{R} \end{cases}$$

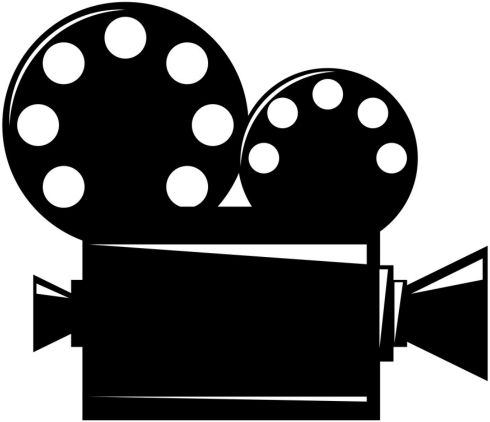
as $n \to \infty$.



Generalisation in higher dimension is straightforward:



$$\begin{cases} \partial_t v = d\Delta v & t > 0, \quad (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & \quad (x,y) \in \mathbb{R}^2 \end{cases}$$



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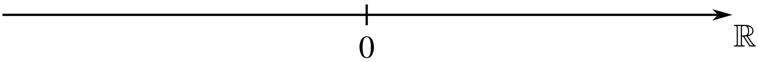

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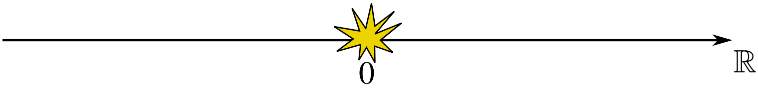

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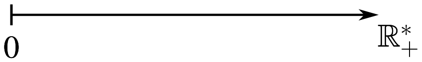

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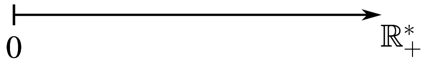

What may happen at $x = 0$?

$$\begin{cases} \partial_t v = d\Delta v & t > 0, \quad x \in \mathbb{R}_+^* \\ \text{Individuals behavior} & x = 0 \\ v|_{t=0} = v_0 & x \in \mathbb{R}_+^* \end{cases}$$

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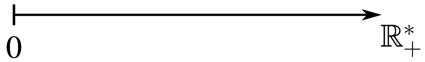

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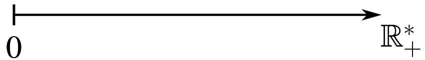

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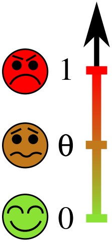

All individuals die (**Dirichlet**): $v|_{x=0} = 0$

Robin: $\theta v|_{x=0} - (1-\theta)\, d\partial_x v|_{x=0} = 0$

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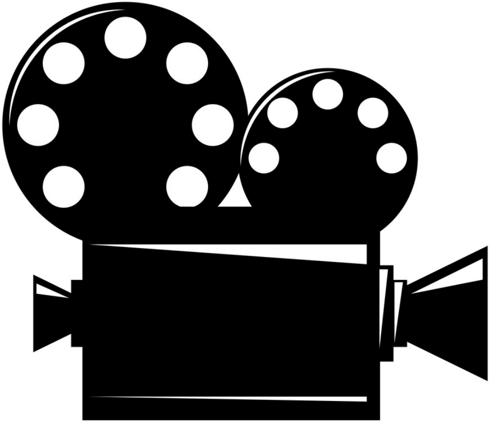

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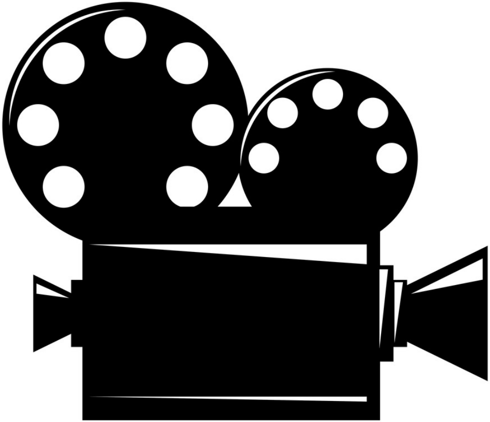

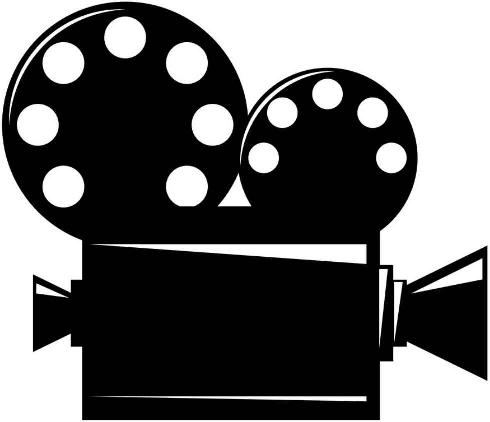

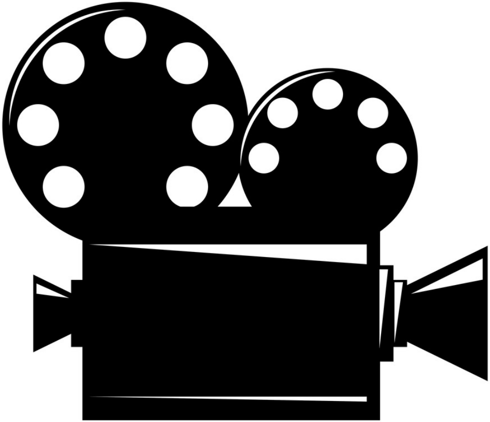

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2013

- Introduction
- Well-posedness
- Comparison principle
- KPP-reaction in the field
- Reaction and drift on the road

{ Berestycki, Coulon, Roquejoffre, Rossi

2015

- Non-local diffusion on the road { Berestycki, Coulon, Roquejoffre, Rossi
- $\mu = \mu(x)$, $\nu = \nu(x)$ periodic { Giletti
- Long range exchanges { Pauthier

2016

- Shape of the solution
- Travelling waves

{ Berestycki, Roquejoffre, Rossi

- In the strip $\mathbb{R} \times]0;L[$ { Tellini

2017

- In the cylinder $\mathbb{R} \times \mathcal{B}_N(0,L)$ { Rossi, Tellini, Valdinoci

2018

- Principal eigen value { Berestycki, Ducasse, Rossi
- Conical domain { Ducasse

2019

- Ecological niche facing climate change { Berestycki, Ducasse, Rossi

2020

- Periodic reaction in the x-direction { Affili

2021

- General cylindrical domain { Bogosel, Giletti, Tellini
- Periodic reaction in the x-direction { Zhang

Heat EQ in the whole space

$$\begin{cases} \partial_t v = d\partial_{xx} v & t > 0, \quad x \in \mathbb{R} \\ v|_{t=0} = v_0 & x \in \mathbb{R} \end{cases}$$

Fourier transform on the variable x

$$\mathcal{F}\left[v(t,\bullet)\right](\xi) = \widehat{v}(t,\xi) := \int_{\mathbb{R}} v(t,x)\, e^{-i\xi x} dx$$

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(variable is t)
(d and ξ are parameters)

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Solve

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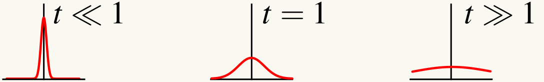

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$\mathcal{F}^{-1}$

$$v(t,x) = \int_{\mathbb{R}} G(t,x-z)\, v_0(z)\, dz$$

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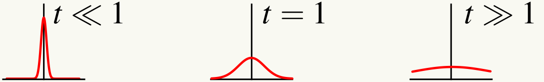

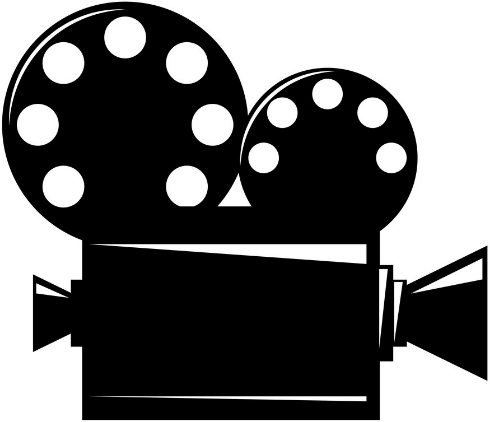

Heat EQ in the half space (continuation approach)

$$\begin{cases} \partial_t v = d\partial_{xx} v & t>0, \quad x>0 \\ \theta v|_{x=0} - (1-\theta)\, d\partial_x v|_{x=0} = 0 & t>0, \quad x=0 \\ v|_{t=0} = v_0 & x>0 \end{cases}$$

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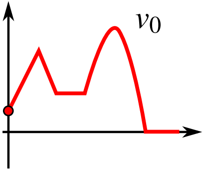

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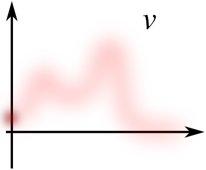

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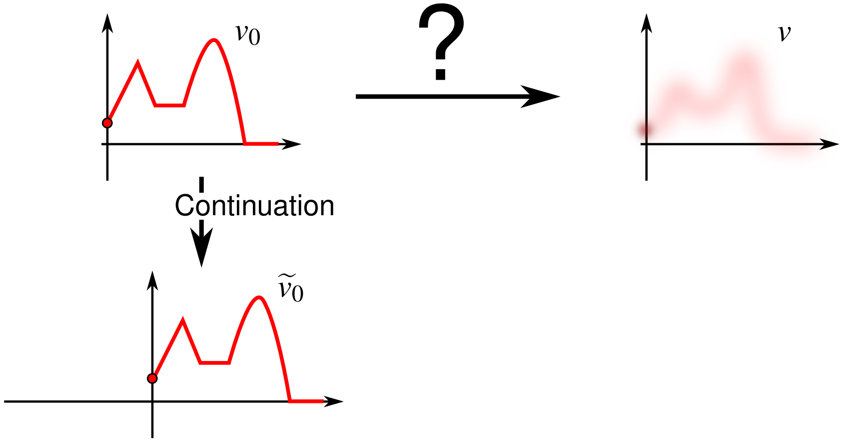

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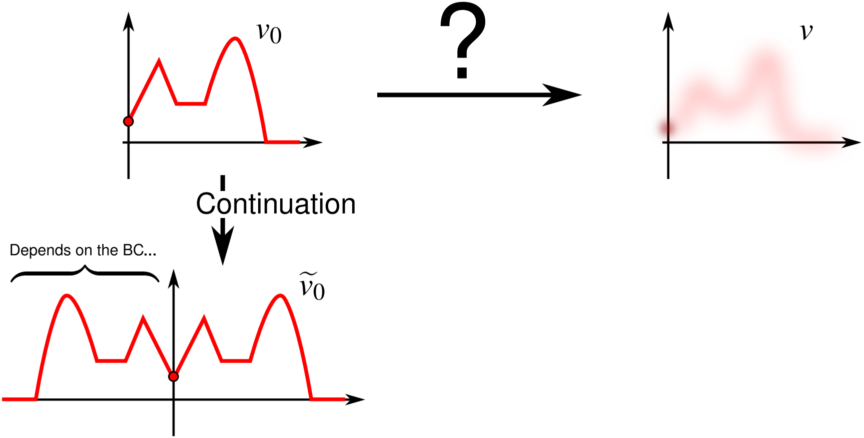

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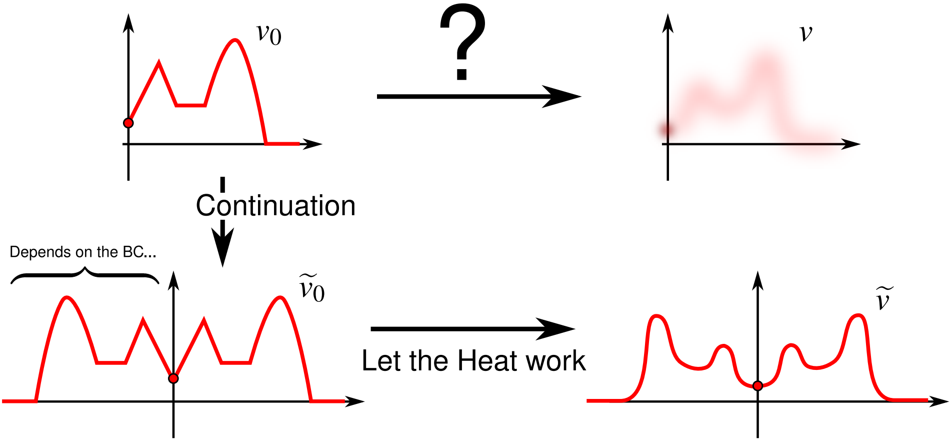

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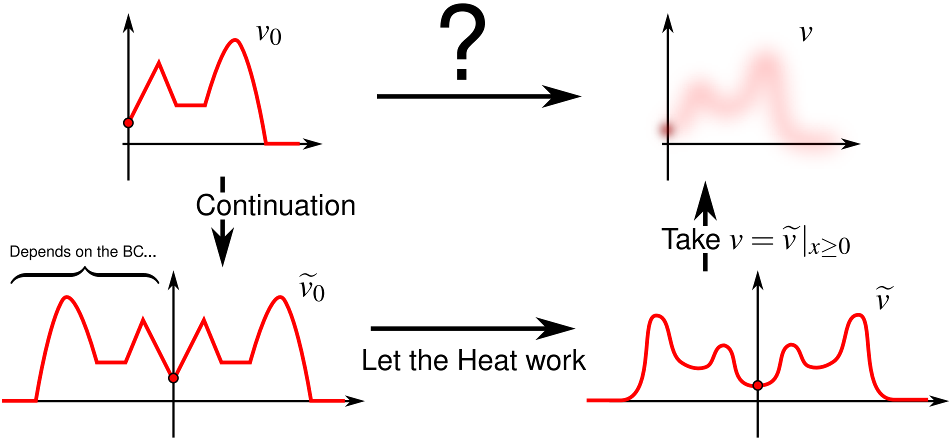

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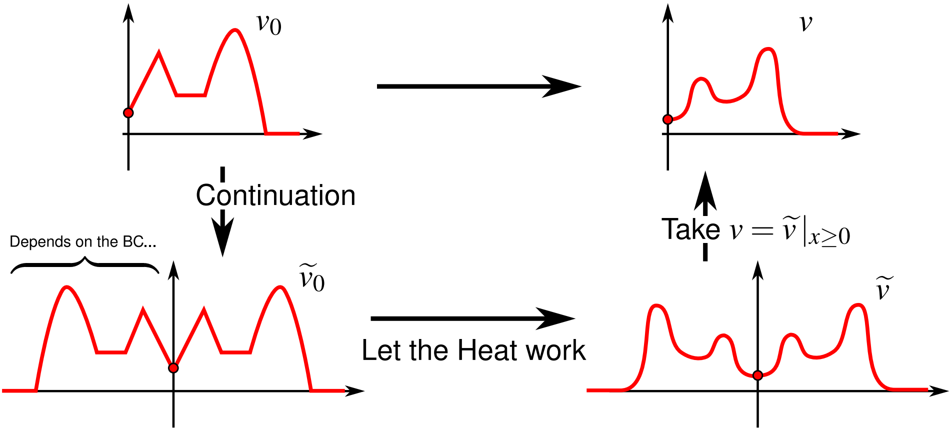

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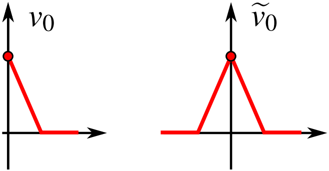

Even continuation

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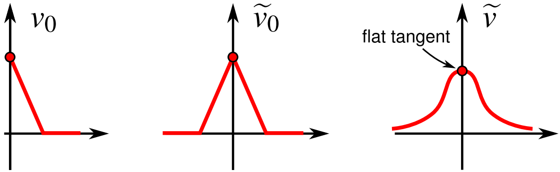

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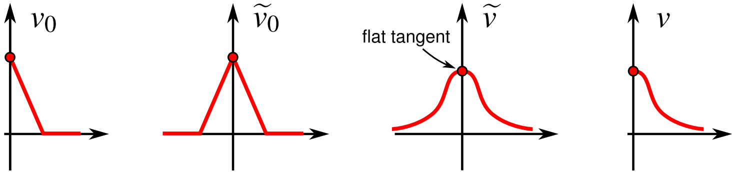
Neumann



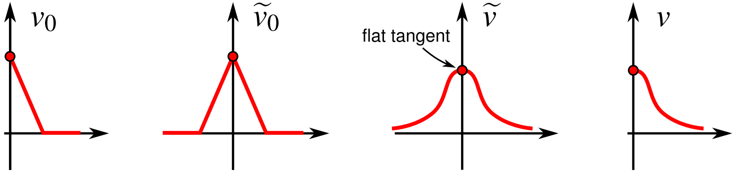
Even continuation

Dirichlet

Heat EQ in the half space (continuation approach)

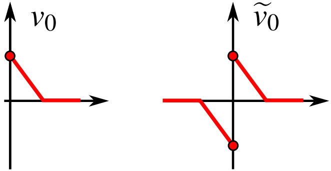
$$
\begin{cases}
\partial_t v = d\partial_{xx} v & t>0, \quad x>0 \\
\theta v|_{x=0} - (1-\theta)\, d\partial_x v|_{x=0} = 0 & t>0, \quad x=0 \\
v|_{t=0} = v_0 & \qquad\quad x>0
\end{cases}
$$

Neumann



Even continuation

Dirichlet

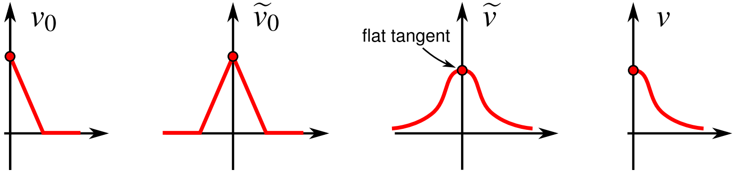


Odd continuation

Heat EQ in the half space (continuation approach)

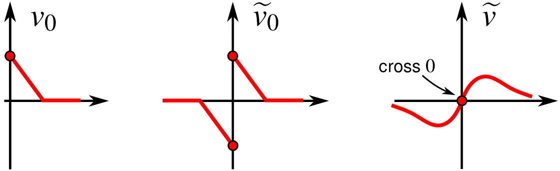
$$\begin{cases} \partial_t v = d\partial_{xx} v & t>0, \quad x>0 \\ \theta v|_{x=0} - (1-\theta)\, d\partial_x v|_{x=0} = 0 & t>0, \quad x=0 \\ v|_{t=0} = v_0 & \quad\quad\quad\;\; x>0 \end{cases}$$

Neumann



Even continuation

Dirichlet



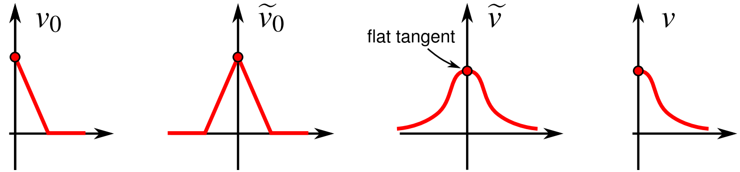
Odd continuation

The field-road diffusion model

Heat EQ in the half space (continuation approach)

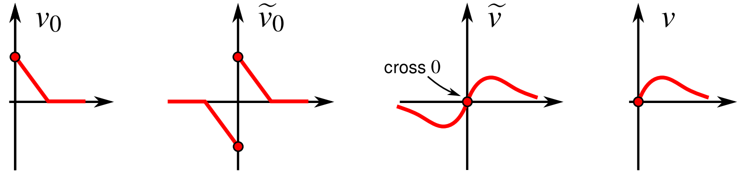
$$
\begin{cases}
\partial_t v = d\partial_{xx} v & t>0, \quad x>0 \\
\theta {\color{red} v|_{x=0}} - (1-\theta)\, {\color{green} d\partial_x v|_{x=0}} = 0 & t>0, \quad x=0 \\
v|_{t=0} = v_0 & x>0
\end{cases}
$$

Neumann

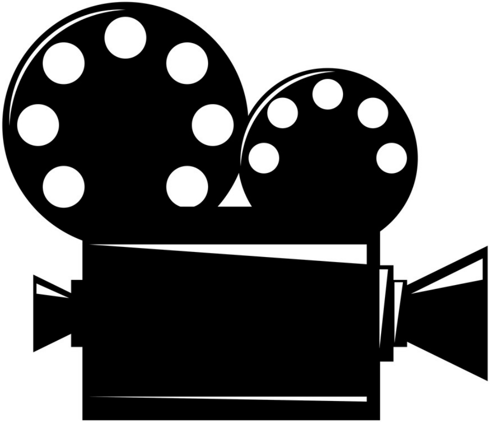


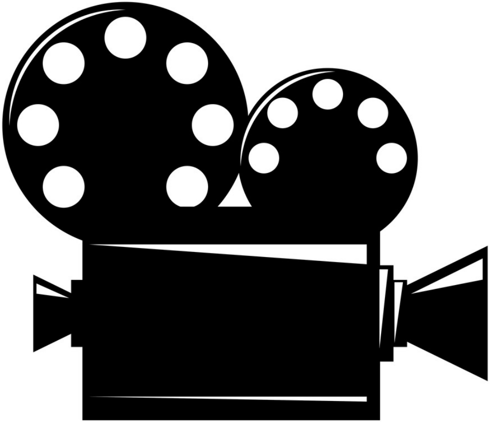
Even continuation

Dirichlet



Odd continuation





Heat EQ in the half space (continuation approach)

$$(*)\begin{cases} \partial_t v = d\partial_{xx} v & t>0, \quad x>0 \\ \theta v|_{x=0} - (1-\theta)\, d\partial_x v|_{x=0} = 0 & t>0, \quad x=0 \\ v|_{t=0} = v_0 & x>0 \end{cases}$$

Proposition (Fundamental solution of the half Heat equation)

The solution to the Cauchy problem $(*)$ is given by :

$$v(t,x) = \int_{\mathbb{R}_+} H_\theta(t,x,z)\, v_0(z)\, dz$$

with

$$H_0(t,x,z) = G(t,x-z) + G(t,x+z) \qquad \text{if } \theta = 0$$

$$H_1(t,x,z) = G(t,x-z) - G(t,x+z) \qquad \text{if } \theta = 1$$

Heat EQ in the half space (continuation approach)

$$(*)\begin{cases} \partial_t v = d\partial_{xx} v & t>0, \quad x>0 \\ \theta v|_{x=0} - (1-\theta)\, d\partial_x v|_{x=0} = 0 & t>0, \quad x=0 \\ v|_{t=0} = v_0 & x>0 \end{cases}$$

Proposition (Fundamental solution of the half Heat equation)

The solution to the Cauchy problem $(*)$ is given by :

$$v(t,x) = \int_{\mathbb{R}_+} H_\theta(t,x,z)\, v_0(z)\, dz$$

with

$$H_\theta(t,x,z) = H_1(t,x,z) + 2G(t,x+z)\left(1 - A\sqrt{\pi d t}\,\frac{\text{Erfc}}{\Gamma}\left(\frac{2Adt+x+z}{2\sqrt{dt}}\right)\right)$$

where $A = \frac{\theta}{d(1-\theta)}$, $\Gamma(\ell) = e^{-\ell^2}$, $\text{Erfc}(\ell) = \frac{2}{\sqrt{\pi}}\int_\ell^{+\infty} e^{-z^2} dz$, if $0<\theta<1$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t > 0, \quad x \in \mathbb{R}, \quad y > 0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t > 0, \quad x \in \mathbb{R}, \quad y = 0 \\ v|_{t=0} = v_0 & x \in \mathbb{R}, \quad y > 0 \end{cases}$$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & \quad x\in\mathbb{R}, \quad y>0 \end{cases}$$

Fourier transform on the variable x

$$\mathcal{F}\left[v(t,\bullet,y)\right](\xi) = \hat{v}(t,\xi,y) := \int_{\mathbb{R}} v(t,x,y)\, e^{-i\xi x}\, dx$$

Breaks ∂_{xx}: $\widehat{\partial_{xx} v(t,\xi,y)} = -\,\xi^2\, \hat{v}(t,\xi,y)$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & \qquad x\in\mathbb{R}, \quad y>0 \end{cases}$$

Fourier transform on the variable x

$$\mathcal{F}\,[v\,(t,\bullet,y)]\,(\xi) = \hat{v}(t,\xi,y) := \int_{\mathbb{R}} v\,(t,x,y)\,e^{-i\xi x}\,dx$$

Breaks ∂_{xx}:
$$\widehat{\partial_{xx} v\,(t,\xi,y)} = -\,\xi^2\,\hat{v}\,(t,\xi,y)$$

Laplace transform on the variable t

$$\mathcal{L}\,[v\,(\bullet,x,y)]\,(s) = \widehat{v}(s,x,y) := \int_0^{+\infty} v\,(t,x,y)\,e^{-st}dt$$

Breaks ∂_t:
$$\widehat{\partial_t v\,(s,x,y)} = s\,\widehat{v}\,(s,x,y) - v_0(x,y)$$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) & t > 0, \quad x \in \mathbb{R}, \quad y > 0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t > 0, \quad x \in \mathbb{R}, \quad y = 0 \\ v|_{t=0} = v_0 & \quad x \in \mathbb{R}, \quad y > 0 \end{cases}$$

x-Fourier/t-Laplace transform

$$\mathcal{F}\mathcal{L}\left[v(\bullet,\bullet,y)\right](s,\xi) = \widehat{v}(s,\xi,y) := \int_{\mathbb{R}} \int_0^{+\infty} v(t,x,y)\, e^{-(st+i\xi x)} dt\, dx$$

Breaks ∂_t and ∂_{xx} !

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & \qquad\quad x\in\mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) \xrightarrow{\boxed{\mathcal{FL}}} \widehat{\partial_t v} = \widehat{d\,(\partial_{xx} v + \partial_{yy} v)}$$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & \quad x\in\mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) \xrightarrow{\ \mathcal{FL}\ } \widehat{\partial_t v} = \widehat{d\,(\partial_{xx} v + \partial_{yy} v)}$$

$\mathcal{FL}$ properties

$$d\partial_{yy}\widehat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\widehat{v}(s,\xi,y) = -\widehat{v_0}\,(\xi,y)$$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) & t > 0, \quad x \in \mathbb{R}, \quad y > 0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t > 0, \quad x \in \mathbb{R}, \quad y = 0 \\ v|_{t=0} = v_0 & x \in \mathbb{R}, \quad y > 0 \end{cases}$$

$$\partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) \xrightarrow{\mathcal{FL}} \widehat{\partial_t v} = \widehat{d\left(\partial_{xx} v + \partial_{yy} v\right)}$$

$\mathcal{FL}$ properties

$$\underbrace{d\partial_{yy}\widehat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\widehat{v}(s,\xi,y) = -\widehat{v_0}\,(\xi,y)}$$

Linear 2^{nd} order ODE
(variable is y)
(d, s and ξ play as parameters)

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & x\in\mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) \xrightarrow{\mathcal{FL}} \widehat{\partial_t v} = \widehat{d\left(\partial_{xx} v + \partial_{yy} v\right)}$$

$\mathcal{FL}$ properties

$$d\partial_{yy}\hat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\hat{v}(s,\xi,y) = -\hat{v}_0(\xi,y)$$

Linear 2nd order ODE

Solve

$$\hat{v}(s,\xi,y) = e^{\frac{\sigma}{\sqrt{d}}y}\left(C_1 - \frac{1}{2\sqrt{d}\sigma}\int_0^y e^{-\frac{\sigma}{\sqrt{d}}\omega}\hat{v}_0(\xi,\omega)\,d\omega\right) + e^{-\frac{\sigma}{\sqrt{d}}y}\left(C_2 + \frac{1}{2\sqrt{d}\sigma}\int_0^y e^{\frac{\sigma}{\sqrt{d}}\omega}\hat{v}_0(\xi,\omega)\,d\omega\right)$$

$\sigma = \sqrt{s + d\xi^2}$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) & t>0, \quad x \in \mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x \in \mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & x \in \mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) \xrightarrow{\mathcal{FL}} \widehat{\partial_t v} = \widehat{d\left(\partial_{xx} v + \partial_{yy} v\right)}$$

$\mathcal{FL}$ properties

$$\underbrace{d\partial_{yy}\hat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\hat{v}(s,\xi,y) = -\hat{v}_0(\xi,y)}_{\text{Linear 2}^{\text{nd}}\text{ order ODE}}$$

Solve

C_1 C_2 Constants *wrt.* y to be determined

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & \quad x\in\mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) \xrightarrow{\mathcal{FL}} \widehat{\partial_t v} = \widehat{d\,(\partial_{xx} v + \partial_{yy} v)}$$

$\mathcal{FL}$ properties

$$\underbrace{d\partial_{yy}\hat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\hat{v}(s,\xi,y) = -\hat{v}_0\,(\xi,y)}_{\text{Linear 2}^{\text{nd}}\text{ order ODE}}$$

Solve

$\hat{v} = 0$ as $y \to +\infty$ (get rid of Tykhonov solutions) → C_1

Constants *wrt.* y to be determined

C_2 ← Robin BC at $y = 0$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & x\in\mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\,(\partial_{xx} v + \partial_{yy} v) \xrightarrow{\mathcal{FL}} \widehat{\partial_t v} = \widehat{d\,(\partial_{xx} v + \partial_{yy} v)}$$

$\mathcal{FL}$ properties

$$d\partial_{yy}\hat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\hat{v}(s,\xi,y) = -\hat{v_0}\,(\xi,y)$$

Linear 2^{nd} order ODE

Solve

$$\hat{v}(s,\xi,y) = \frac{1}{2\sqrt{d}} \int_0^{+\infty} \left(\frac{e^{-\frac{\sigma}{\sqrt{d}}|y-\omega|}}{\sigma} + \frac{e^{-\frac{\sigma}{\sqrt{d}}(y+\omega)}}{\sigma} - 2A\sqrt{d}\,\frac{e^{-\frac{\sigma}{\sqrt{d}}(y+\omega)}}{\sigma\left(\sigma + A\sqrt{d}\right)} \right) \hat{v_0}\,(\xi,\omega)\, d\omega$$

$\sigma = \sqrt{s + d\xi^2}$

Heat EQ in the half space (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ \theta v|_{y=0} - (1-\theta)\, d\partial_x v|_{y=0} = 0 & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ v|_{t=0} = v_0 & \quad x\in\mathbb{R}, \quad y>0 \end{cases}$$

$$\partial_t v = d\left(\partial_{xx} v + \partial_{yy} v\right) \xrightarrow{\mathcal{FL}} \widehat{\partial_t v} = \widehat{d\left(\partial_{xx} v + \partial_{yy} v\right)}$$

$\mathcal{FL}$ properties

$$\underbrace{d\partial_{yy}\hat{v}(s,\xi,y) - \left(s + d\,\xi^2\right)\hat{v}(s,\xi,y) = -\hat{v}_0(\xi,y)}_{\text{Linear } 2^{\text{nd}} \text{ order ODE}}$$

“Only” remains then to take the inverse Fourier/Laplace transform...

$$v(t,x,y) = \int_{\mathbb{R}}\int_{\mathbb{R}_+} H_\theta(t,x,y,z,\omega)\, v_0(z,\omega)\, dz\, d\omega$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, & x\in\mathbb{R} \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, & x\in\mathbb{R}. \end{cases}$$

$$\begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, & x\in\mathbb{R} \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, & x\in\mathbb{R}. \end{cases}$$

$$\begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Theorem (Alfaro, Ducasse, Tréton, 22')
(Solution of the field-road diffusive model)

The solution to the latter Cauchy problem is

$$v(t,x,y) = V(t,x,y) + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}} \Lambda(t,z,y)\, u_0(x-z)\, dz + \frac{\mu\nu}{\sqrt{d}}\int_0^t\int_{\mathbb{R}} \Lambda(s,z,y)\, V|_{y=0}(t-s,x-z)\, dz ds,$$

$$u(t,x) = e^{-\mu t}U(t,x) + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}} G(t-s,x-z)\, v|_{y=0}(s,z)\, dz ds$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, \quad (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, \quad x\in\mathbb{R} \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, \quad x\in\mathbb{R}. \end{cases} \qquad \begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Theorem (Alfaro, Ducasse, Tréton, 22')
(Solution of the field-road diffusive model)

The solution to the latter Cauchy problem is

$$v(t,x,y) = \boxed{V(t,x,y)} + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}} \Lambda(t,z,y)\, u_0(x-z)\, dz + \frac{\mu\nu}{\sqrt{d}}\int_0^t\int_{\mathbb{R}} \Lambda(s,z,y)\, \boxed{V|_{y=0}(t-s,x-z)}\, dz\, ds,$$

$$u(t,x) = e^{-\mu t}U(t,x) + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}} G(t-s,x-z)\, v|_{y=0}(s,z)\, dz\, ds$$

where $V = V(t,X)$ is the solution to the Cauchy problem

$$\begin{cases} \partial_t V = d\Delta V, & t>0, \quad x\in\mathbb{R}, \quad y>0, \\ \nu V|_{y=0} - d\partial_y V|_{y=0} = 0, & t>0, \quad x\in\mathbb{R}, \\ V|_{t=0} = v_0, & x\in\mathbb{R}, \quad y>0, \end{cases}$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, \quad (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, \quad x\in\mathbb{R} \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, \quad x\in\mathbb{R}. \end{cases} \qquad \begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Theorem (Alfaro, Ducasse, Tréton, 22')
(Solution of the field-road diffusive model)

The solution to the latter Cauchy problem is

$$v(t,x,y) = V(t,x,y) + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}} \Lambda(t,z,y)\, u_0(x-z)\, dz + \frac{\mu\nu}{\sqrt{d}}\int_0^t \int_{\mathbb{R}} \Lambda(s,z,y)\, V|_{y=0}(t-s,x-z)\, dz\,ds,$$

$$u(t,x) = e^{-\mu t}\boxed{U(t,x)} + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}} G(t-s,x-z)\, v|_{y=0}(s,z)\, dz\,ds$$

where $U = U(t,x)$ is the solution to the Cauchy problem

$$\begin{cases} \partial_t U = D\partial_{xx}U, & t>0, \quad x\in\mathbb{R}, \\ U|_{t=0} = u_0, & x\in\mathbb{R}, \end{cases}$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, \quad (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, \quad x\in\mathbb{R} \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, \quad x\in\mathbb{R}. \end{cases} \qquad \begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Theorem (Alfaro, Ducasse, Tréton, 22')
(Solution of the field-road diffusive model)

The solution to the latter Cauchy problem is

$$v(t,x,y) = V(t,x,y) + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}} \Lambda(t,z,y)\, u_0(x-z)\, dz + \frac{\mu\nu}{\sqrt{d}}\int_0^t \int_{\mathbb{R}} \Lambda(s,z,y)\, V|_{y=0}(t-s,x-z)\, dz ds,$$

$$u(t,x) = e^{-\mu t}U(t,x) + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}} \boxed{G(t-s,x-z)}\, v|_{y=0}(s,z)\, dz ds$$

where

$G = G(t,x)$ denotes the one-dimensional D-diffusive Heat-kernel :

$$G(t,x) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}},$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, \quad (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, \quad x\in\mathbb{R} \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, \quad x\in\mathbb{R}. \end{cases} \qquad \begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Theorem (Alfaro, Ducasse, Tréton, 22')
(Solution of the field-road diffusive model)

The solution to the latter Cauchy problem is

$$v(t,x,y) = V(t,x,y) + \frac{\mu}{\sqrt{d}}\int_{\mathbb{R}} \Lambda(t,z,y)\, u_0(x-z)\, dz + \frac{\mu\nu}{\sqrt{d}}\int_0^t\int_{\mathbb{R}} \Lambda(s,z,y)\, V|_{y=0}(t-s,x-z)\, dz\, ds,$$

$$u(t,x) = e^{-\mu t}U(t,x) + \nu\int_0^t e^{-\mu(t-s)}\int_{\mathbb{R}} G(t-s,x-z)\, v|_{y=0}(s,z)\, dz\, ds$$

where

$$\Lambda(t,x,y) = \frac{e^{-\frac{y^2}{4dt}}}{2\pi}\int_{\mathbb{R}^{N-1}} \left[a\alpha\Phi_\alpha + b\beta\Phi_\beta + c\gamma\Phi_\gamma\right](t,\xi,y)\, e^{-dt\xi^2+i\xi x}\, d\xi,$$

Heat on the field-road model (Fourier/Laplace approach)

$$\begin{cases} \partial_t v = d\Delta v & t>0, \quad (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, \quad x\in\mathbb{R} \\ \partial_t u = D\partial_{xx} u + \nu v|_{y=0} - \mu u & t>0, \quad x\in\mathbb{R}. \end{cases} \qquad \begin{cases} v|_{t=0} = v_0 & (x,y)\in\mathbb{R}\times\mathbb{R}_+^* \\ u|_{t=0} = u_0 & x\in\mathbb{R}. \end{cases}$$

Theorem (Alfaro, Ducasse, Tréton, 22')
(Solution of the field-road diffusive model)

$$\Lambda(t,x,y) = \frac{e^{-\frac{y^2}{4dt}}}{2\pi}\int_{\mathbb{R}^{N-1}} \left[a\alpha\Phi_\alpha + b\beta\Phi_\beta + c\gamma\Phi_\gamma\right](t,\xi,y)\, e^{-dt\xi^2 + i\xi x}\, d\xi,$$

- $(\alpha,\beta,\gamma) = (\alpha,\beta,\gamma)(\xi)$ being the three complex roots of the δ-indexed polynomials

$$P_\delta(\sigma) = \sigma^3 + \frac{\nu}{\sqrt{d}}\sigma^2 + (\mu+\delta)\,\sigma + \frac{\nu\,\delta}{\sqrt{d}}, \qquad \text{with } \delta = (D-d)\,\xi^2,$$

- $a = \dfrac{1}{(\alpha-\beta)(\alpha-\gamma)}, \qquad b = \dfrac{1}{(\beta-\alpha)(\beta-\gamma)}, \qquad c = \dfrac{1}{(\gamma-\alpha)(\gamma-\beta)},$

- $\Phi_\bullet(t,\xi,y) = \dfrac{\text{Erfc}}{\Gamma}\left(\dfrac{-2\bullet\sqrt{d}\,t + y}{2\sqrt{dt}}\right)$

- Diffusion provokes extinction of the population by spreading

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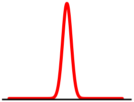

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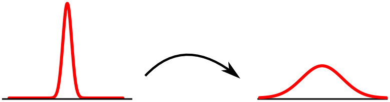

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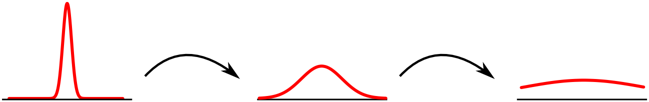

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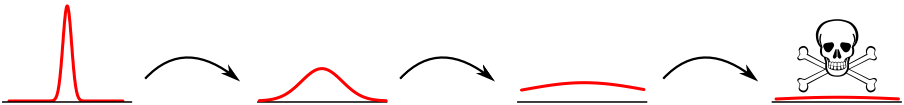
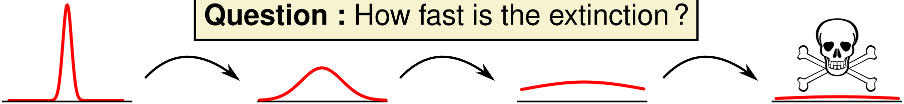


- Diffusion provokes extinction of the population by spreading

Question : How fast is the extinction ?



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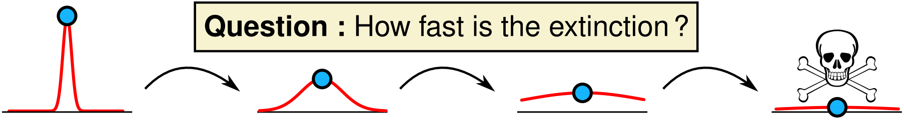
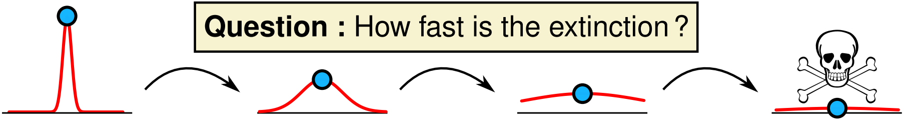


- We look at the max in space

The field-road diffusion model

- Diffusion provokes extinction of the population by spreading

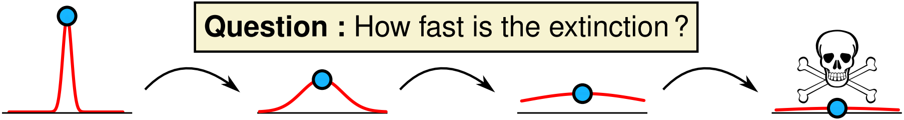
Question : How fast is the extinction ?



- We look at the $\max$ in space and expect that

$$\|v(t,\cdot)\|_{L^\infty} \leq \frac{c}{t^k}$$

- Diffusion provokes extinction of the population by spreading



Question : How fast is the extinction ?

- We look at the $\max$ in space and expect that

$$\|v(t,\cdot)\|_{L^\infty} \leq \frac{c}{t^k}$$

 Large k means **aggressive** diffusion

 Small k means **nice** diffusion

- Classical example:

$$\begin{cases} \partial_t v = d\Delta v & t > 0, \quad (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & \quad (x,y) \in \mathbb{R}^2 \end{cases}$$

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$$|v(t,x,y)| = |G(t,\bullet) * v_0|(x,y)$$

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$$|v(t,x,y)| = |G(t,\bullet) * v_0|(x,y) = \left| \frac{1}{4\pi dt} e^{-\frac{\|\bullet\|^2}{4dt}} * v_0 \right| (x,y)$$

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$$\leq \left\|\frac{1}{4\pi dt}e^{-\frac{\|\bullet\|^2}{4dt}}\right\|_{L^\infty}\times\|v_0\|_{L^1}$$

$$\|v(t,\cdot)\|_{L^\infty} \leq \frac{c\left(d,\|v_0\|_{L^1}\right)}{t^{1}}$$

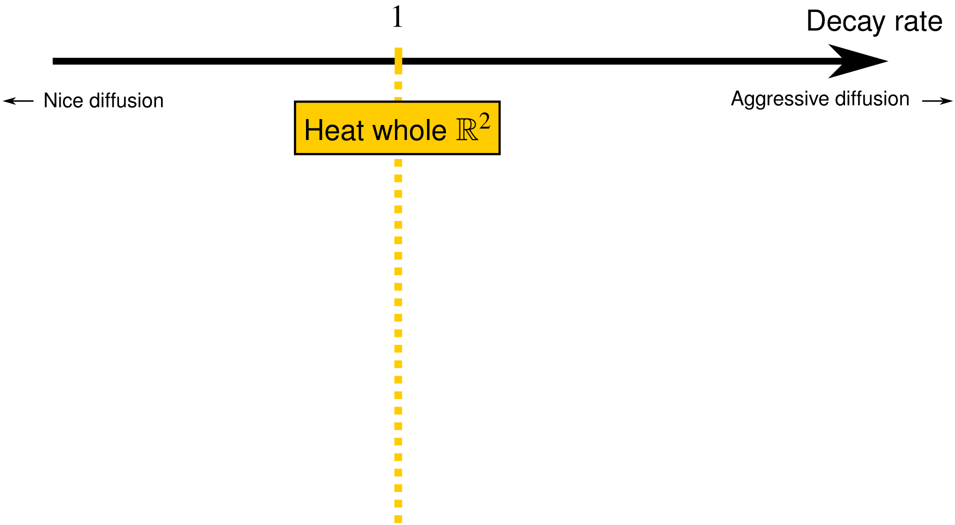
Outcomes for our problems :

Decay rate →

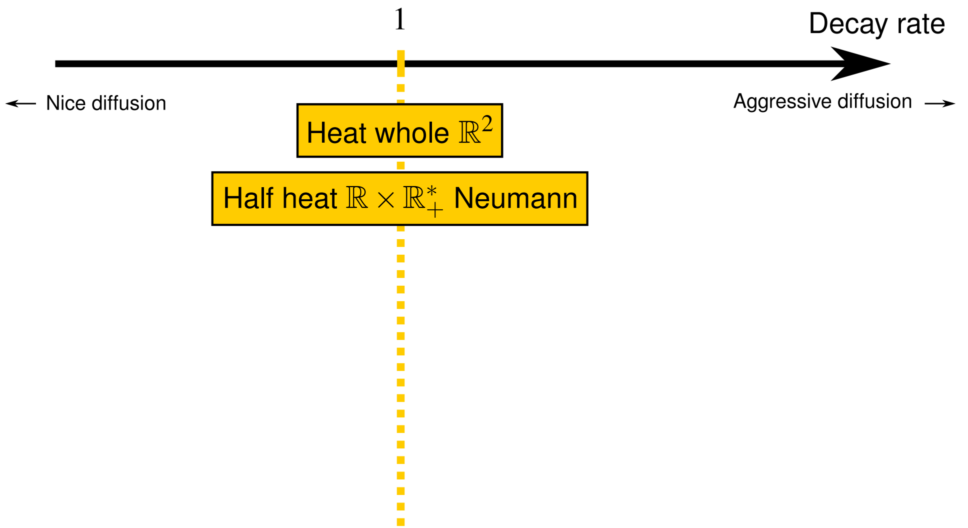
← Nice diffusion

Aggressive diffusion →

- Outcomes for our problems :

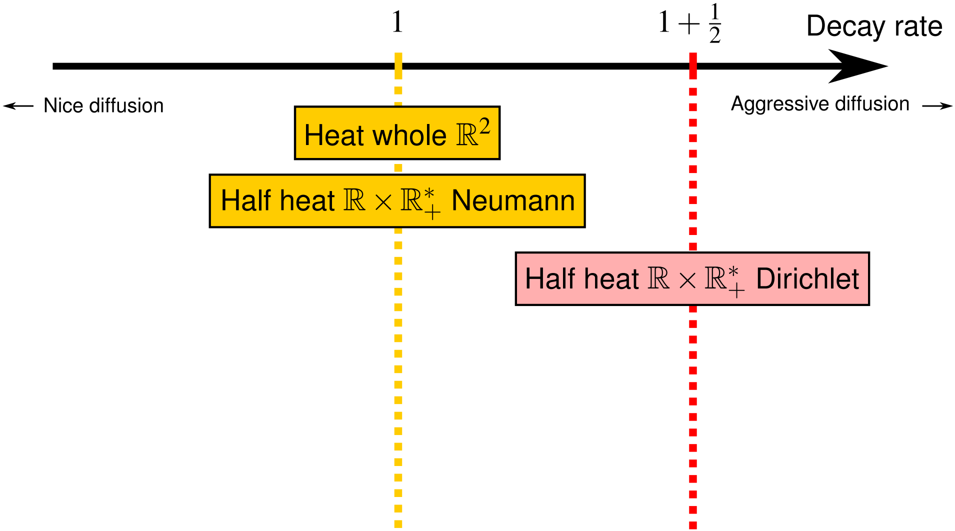


- Outcomes for our problems :

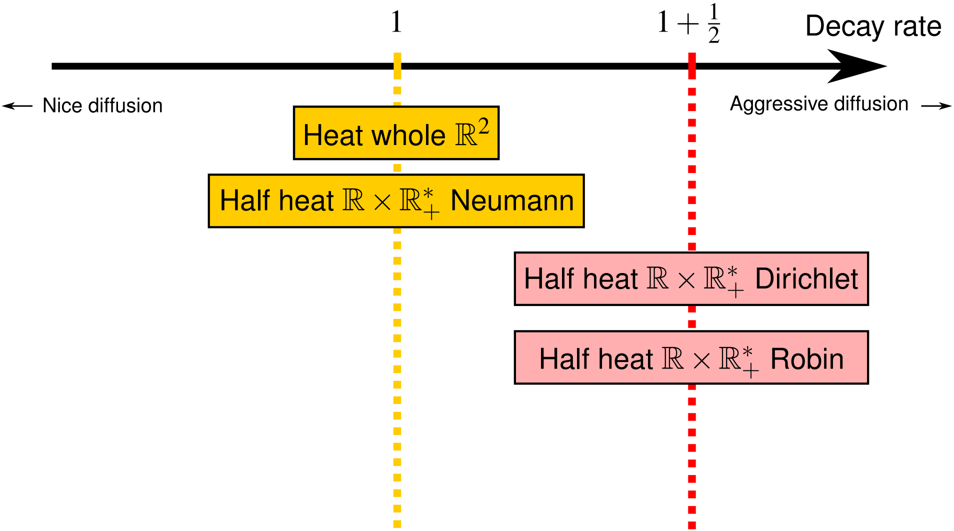


The field-road diffusion model

Outcomes for our problems :



Outcomes for our problems :



The field-road diffusion model

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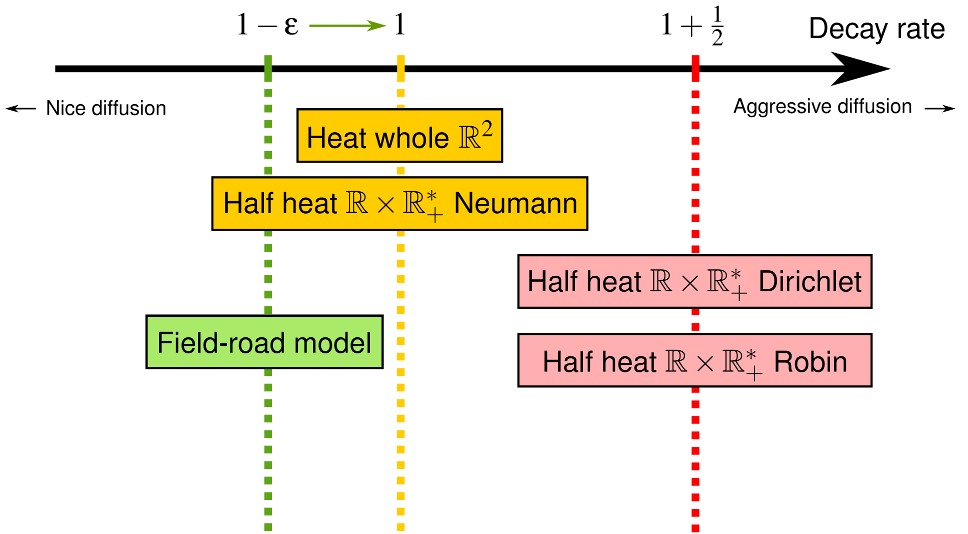

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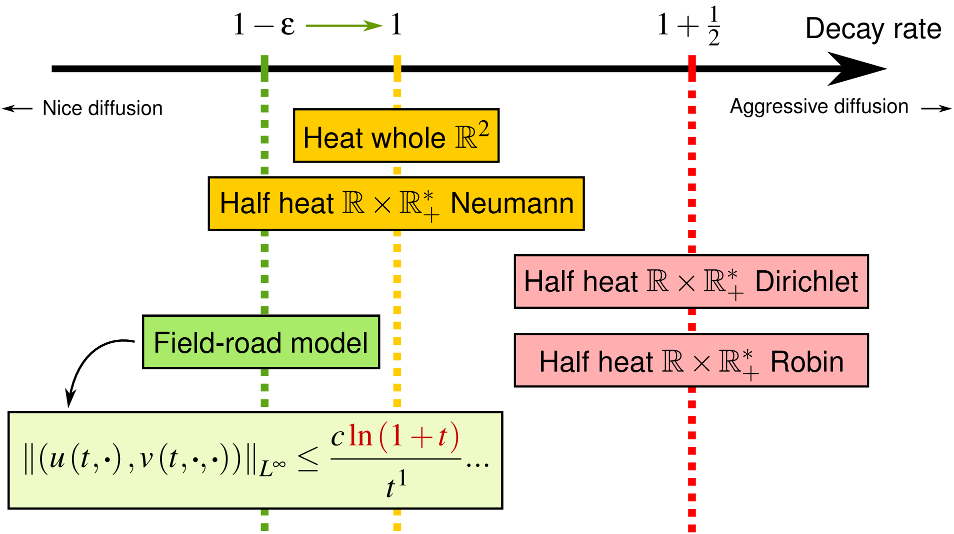
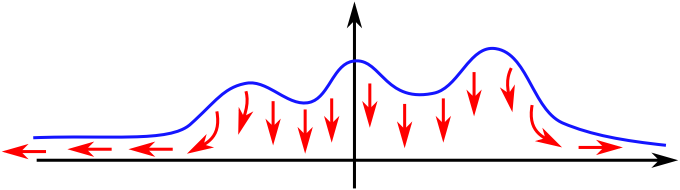


- Adding births and deaths

Diffusion

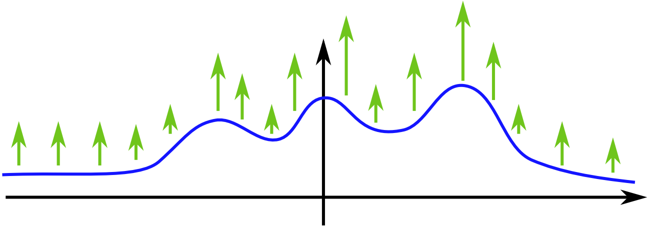
$$\begin{cases} \partial_t v = d\Delta v + f(v) & t > 0, \quad (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & (x,y) \in \mathbb{R}^2 \end{cases}$$



- Adding births and deaths

Reaction

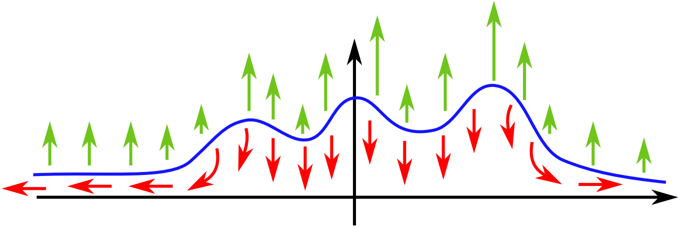
$$\begin{cases} \partial_t v = d\Delta v + \boxed{f(v)} & t > 0, \quad (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & (x,y) \in \mathbb{R}^2 \end{cases}$$



- Adding births and deaths

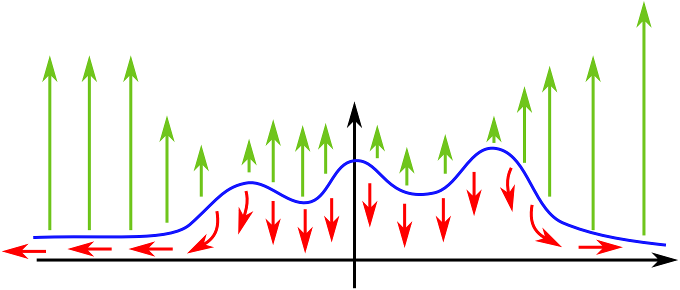
Diffusion **VS** Reaction

$$\begin{cases} \partial_t v = d\Delta v + f(v) & t > 0, \quad (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & (x,y) \in \mathbb{R}^2 \end{cases}$$



Question : Persistence or extinction ?

- Adding births and deaths $\begin{cases} \partial_t v = d\Delta v + f(v) & t > 0, \ (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & (x,y) \in \mathbb{R}^2 \end{cases}$



Logistic $f(v) = v(1-v)$

- Adding births and deaths
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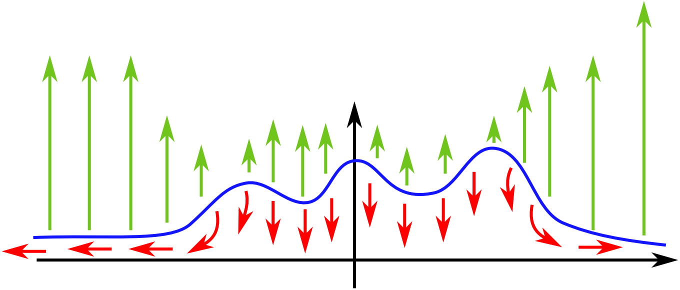

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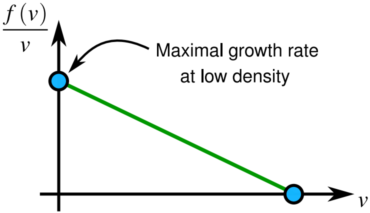

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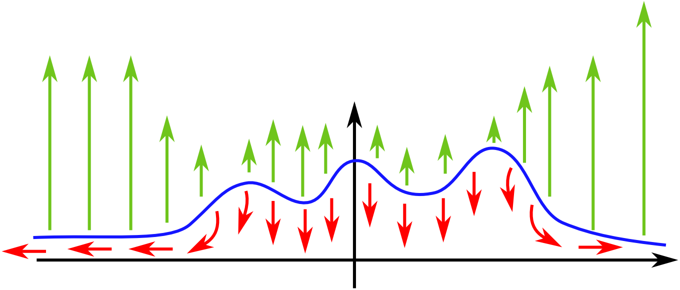
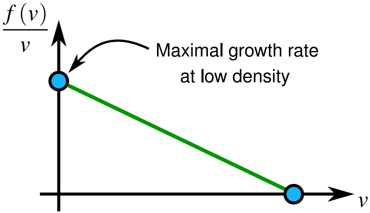


Logistic $f(v) = v(1-v)$

Reaction always wins:

Systematic invasion



➢ Adding births and deaths
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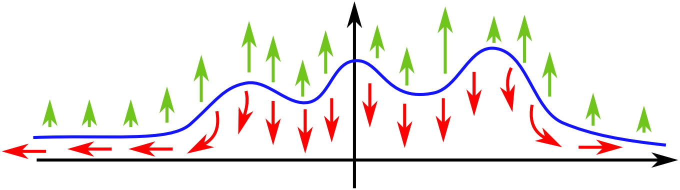

Monostable $f(v) = v^{1+p}(1-v)$

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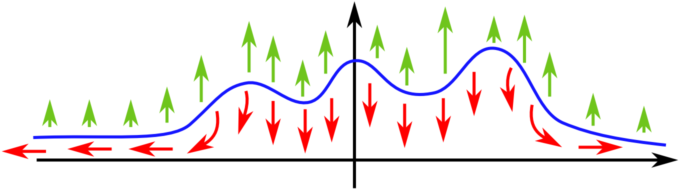
Allee effect, *ie* harder to grow at low density



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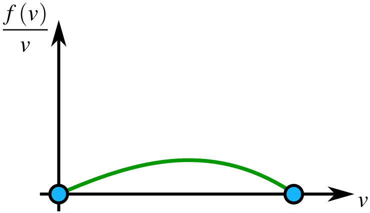

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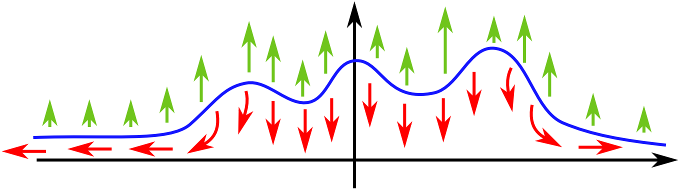
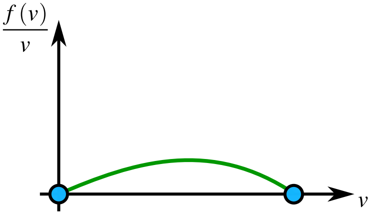


Monostable $f(v) = v^{1+p}(1-v)$

Reaction may lose...



$$\begin{cases} \partial_t v = d\Delta v + v^{1+p}(1-v) & t > 0, \quad (x,y) \in \mathbb{R}^2 \\ v|_{t=0} = v_0 & \quad (x,y) \in \mathbb{R}^2 \end{cases}$$

Theorem (Aronson, Weinberger, 78')

Let $p_F := 1$, then

Soft Allee effect if $0 < p \leq p_F$, the invasion is systematic,

Hard Allee effect if $p > p_F$, there are some “small enough” initial data which shall become extinct

$$\begin{cases} \partial_t v = d\Delta v + v^{1+p}(1-v) & t>0, \quad (x,y)\in\mathbb{R}^2 \\ v|_{t=0} = v_0 & (x,y)\in\mathbb{R}^2 \end{cases}$$

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Hard Allee effect if $p > p_F$, there are some "small enough" initial data which shall become extinct

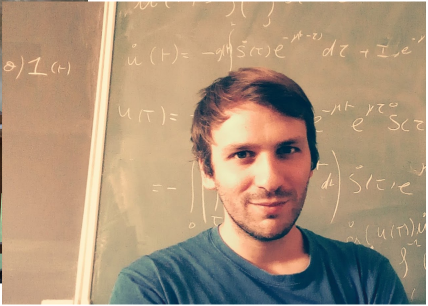
straightly comes from $\|v(t,\cdot)\|_{L^\infty} \leq \dfrac{c}{t^1}$ ...

Open question: can we find such a result for

$$\begin{cases} \partial_t v = d\Delta v + \boxed{v^{1+p}(1-v)} & t>0, \quad x\in\mathbb{R}, \quad y>0 \\ -d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0} & t>0, \quad x\in\mathbb{R}, \quad y=0 \\ \partial_t u = D\partial_{xx}u + \nu v|_{y=0} - \mu u & t>0, \quad x\in\mathbb{R} \end{cases}$$

now we know the decay rate of the diffusive field-road model ?...

Thanks for your attention!



Matthieu Alfaro

Romain Ducasse