Bridging bulk and surface:
An interacting particle system
towards the field-road diffusion model

Matthieu Alfaro[1], Mustapha Mourragui[2] and Samuel Tréton[3]

Abstract

We recover the so-called field-road diffusion model as the hydrodynamic limit of an interacting particle system. The former consists of two parabolic PDEs posed on two sets of different dimensions (a "field" and a "road" in a population dynamics context), and coupled through exchange terms between the field's boundary and the road. The latter stands as a Symmetric Simple Exclusion Process (SSEP): particles evolve on two microscopic lattices following a Markov jump process, with the constraint that each site cannot host more than one particle at the same time. The system is in contact with reservoirs that allow to create or remove particles at the boundary sites. The dynamics of these reservoirs are slowed down compared to the diffusive dynamics, to reach the reactions and the boundary conditions awaited at the macroscopic scale. This issue of bridging two spaces of different dimensions is, as far as we know, new in the hydrodynamic limit context, and raises perspectives towards future related works.

Key words: hydrodynamic limit, interacting particle system, simple exclusion processes, slow boundary reservoirs, field-road diffusion model.

MSC2020: 60J27, 82C22, 35K57, 37N25.

Contents

[1,2,3] Univ. Rouen Normandie, CNRS, Normandie Univ., LMRS UMR 6085, F-76000 Rouen, France.

1   Introduction

The goal of the present work is to derive the *field-road diffusion model* as the hydrodynamic limit of an interacting particle system. The former was introduced by Berestycki, Roquejoffre and Rossi [10] in order to describe spread of diseases or invasive species in presence of networks with accelerated propagation. It consists of two parabolic PDEs posed on two sets of different dimensions (a field and a road in a population dynamics context), and coupled through exchange terms between the field's boundary and the road — see subsection 1.1 for details. To asymptotically retrieve this deterministic model from a stochastic interacting particle system, we consider a Symmetric Simple Exclusion Process (SSEP) which evolves both on a finite discrete cylinder (the field) and its lower boundary (the road). Characterizing the SSEP, the microscopic dynamics is tied with a *simple exclusion rule* that forces each site to host at most one particle at the same time. To manage in particular the coupling between the field and the road, the system is in contact with reservoirs that allow to create or remove particles at the boundary sites of the cylinder. The activity of these reservoirs is slowed down compared to the diffusive dynamics, in order to align with the exchange terms awaited at the macroscopic scale. The originality of our analysis stands in the coupling between two domains of different dimensions, an issue that, as far as we know, has never been considered when recovering diffusive PDEs as the hydrodynamic limit of exclusion processes — see subsection 1.2.

1.1   The field-road model for fast diffusion channels

Recently, there has been a growing recognition of the importance of *fast diffusion channels* on biological invasions: for instance, an accidental transportation via human activities of some individuals towards northern and eastern France may be the cause of accelerated propagation of the pine processionary moth [46]. In Canada, some GPS data revealed that wolves travel faster along seismic lines (i.e. narrow strips cleared for energy exploration), thus increasing their chances to meet a prey [41]. It is also acknowledged that fast diffusion channels (roads, airlines, etc.) play a central role in the propagation of epidemics. As is well known, the spread of the black plague, which killed about a third of the European population in the 14th century, was favoured by the trade routes, especially the Silk Road, see [49]. More recently, some evidences of the the radiation of the COVID epidemic along highways and transportation infrastructures were found [31].

In this context, the *field-road model* introduced by Berestycki, Roquejoffre and Rossi [10] writes as

$$
\begin{cases}
\partial_t v = d\Delta v + f(v), & t > 0, \quad x \in \mathbb{R}^{p-1}, \quad y > 0, \\
-d\partial_y v|_{y=0} = \alpha u - \beta v|_{y=0}, & t > 0, \quad x \in \mathbb{R}^{p-1}, \\
\partial_t u = D\Delta u + \beta v|_{y=0} - \alpha u, & t > 0, \quad x \in \mathbb{R}^{p-1}.
\end{cases}
\tag{1.1}
$$

The mathematical problem then amounts to describing survival and propagation in a nonstandard physical space: the geographical domain consists in the half-space (the "field") $x \in \mathbb{R}^{p-1}$, $y > 0$, bordered by the hyperplane (the "road") $x \in \mathbb{R}^{p-1}$, $y = 0$. In the field, individuals diffuse with coefficient $d > 0$ and their density is given by $v = v(t, x, y)$. In particular Δv has to be understood as $\Delta_x v + \partial_{yy} v$. On the road, individuals typically diffuse faster ($D > d$) and their density is given by $u = u(t, x)$. In particular Δu has to be understood as $\Delta_x u$. The exchanges of population between the road and the field

are described by the second equation in system (1.1), where $\alpha > 0$ and $\beta > 0$. These boundary conditions, and the zeroth-order term on the road, link the field and the road equations and are the core of the model (see also the *volume-surface* systems [18], [26], [23] in the context of chemical processes or asymmetric stem cell division).

In a series of works [10, 9, 11, 12], Berestycki, Roquejoffre and Rossi studied the field-road system with $p = 2$ and f a Fisher-KPP nonlinearity. They shed light on an *acceleration phenomenon*: when $D > 2d$, the road enhances the global diffusion and the spreading speed exceeds the standard Fisher-KPP invasion speed. This new feature has stimulated many works and, since then, many related problems taking into account heterogeneities, more complex geometries, nonlocal diffusions, etc. have been studied [5, 6], [33], [44, 45, 43], [52], [47], [22], [7, 8], [1, 54], [16].

Very recently, the *purely diffusive* field-road system — obtained by letting $f \equiv 0$ in (1.1) — has attracted some attention. Hence, an explicit expression for both the fundamental solution and the solution to the associated Cauchy problem, and a sharp (possibly up to a logarithmic term) decay rate of the L^∞ norm of the solution were obtained in [3]. In a bounded domain, the long time convergence was studied [2] through entropy methods, in both the continuous and the discrete (finite volume scheme) settings.

From now on, we thus consider the purely diffusive field-road model. By using the rescaling

$$\tilde{v}(t,x,y) = v\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}, \frac{y}{\lambda}\right), \quad \tilde{u}(t,x) = \lambda u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right), \quad \lambda = \frac{\alpha}{\beta},$$

we see that it is enough to consider the case $\alpha = \beta$. Also, for $p \geq 2$, we work in the p-dimensional open finite cylinder

$$\Lambda := \mathbb{T}^{p-1} \times (0,1),$$

where $\mathbb{T}$ is the one-dimensional torus $\mathbb{R}/\mathbb{Z}$. For v, we impose the zero Neumann boundary conditions on the upper boundary $\mathbb{T}^{p-1} \times \{y = 1\}$. This insures the conservation of the total mass, namely $\int_{\mathbb{T}^{p-1}\times(0,1)} v(t,x,y)dxdy + \int_{\mathbb{T}^{p-1}} u(t,x)dx$, therefore modeling a purely diffusive process within a closed environment. Denoting n the unit outward normal vector to $\partial\Lambda$, the considered system is thus

$$\begin{cases} \partial_t v = d\Delta v, & t > 0, \quad x \in \mathbb{T}^{p-1}, \quad y \in (0,1), \\ -d\partial_y v|_{y=0} = \alpha u - \alpha v|_{y=0}, & t > 0, \quad x \in \mathbb{T}^{p-1}, \quad y = 0, \\ \partial_t u = D\Delta u + \alpha v|_{y=0} - \alpha u, & t > 0, \quad x \in \mathbb{T}^{p-1}, \\ \frac{\partial v}{\partial n} = 0, & t > 0, \quad x \in \mathbb{T}^{p-1}, \quad y = 1, \end{cases} \tag{1.2}$$

supplemented with an initial condition

$$\begin{cases} v|_{t=0} = v_0 \in L^\infty(\Lambda) \cap [0,1]^\Lambda, & x \in \mathbb{T}^{p-1}, \quad y \in (0,1), \\ u|_{t=0} = u_0 \in L^\infty(\mathbb{T}^{p-1}) \cap [0,1]^{\mathbb{T}^{p-1}}, & x \in \mathbb{T}^{p-1}. \end{cases} \tag{1.3}$$

Note that, given the linear nature of the system (1.2), the use of initial data bounded by 1 is a simplification that does not compromise the generality of our analysis. This choice is actually imposed by the exclusion rule, see Remark 2.3.

1.2   Interacting Particle Systems and Simple Exclusion Processes

The field of interacting particle systems is a branch of probability theory that emerged in the early 1970s, focusing on Markov processes inspired by models from statistical physics and biology. Analysis occurs at both the microscopic level of particle dynamics and by scaling from microscopic to macroscopic levels. This involves space and time renormalization procedures to derive hydrodynamic limits, represented by PDEs that describe key model quantities such as particle densities.

Introduced by Frank Spitzer in [50], the exclusion process are interacting particle systems from which can be recovered a large variety of diffusive systems driven out of equilibrium, see the pioneering works [51], [40, 39]. We refer to the seminal book [34] for the complete derivation of the heat equation on a torus from a nearest-neighbor exclusion process which consists in a collection of continuous-time random walks evolving on a lattice (see below for details).

When boundaries are considered, the system is in contact with some so-called reservoirs, see [36], [19, 21, 20] where Dirichlet boundary conditions are recovered. Recently, a lot of effort has been put in understanding the case of exclusion process whose dynamics is perturbed by the presence of a slow bond [15], [27], or by slow boundary effects [27, 28, 29], [4], [35], [37], [13].

Let us comment more precisely on some of the outcomes obtained in [4]. The authors specifically examine the hydrodynamic behavior of a symmetric simple exclusion process with *slow boundary*. This means that, at the boundary sites, particles can be born or die at slower rates (depending on the scaling parameter N) than events occurring in the bulk. The hydrodynamic limit is then the heat equation, supplemented with Dirichlet, Robin, or Neumann boundary conditions, depending on the scaling of the boundary rates.

The present work stands at the crossroads of this framework, the reaction-diffusion issues and the epidemiology/population dynamics modeling. Regarding this, let us mention the very recent work [42] where a reaction-diffusion system modeling the sterile insect technique is retrieved. The very originality of our work stands in the fact that the considered system is posed on sets of different dimensions a case which, as far as we know, is considered for the first time in the interacting particle system literature.

2   Notations and main result

All the notations used in this paper are gathered in the Table of Notations at the end of this document.

2.1   Sets and related notations

As announced above, in the macroscopic setting, we work in the p-dimensional open finite cylinder

$$\Lambda := \mathbb{T}^{p-1} \times (0,1),$$

equipped with its Borel σ-algebra, where $\mathbb{T}$ designates the one-dimensional torus $\mathbb{R}/\mathbb{Z}$. The boundary of the domain is denoted

$$\Gamma := \partial\Lambda = \{(x,y) \in \overline{\Lambda} \mid y = 0 \text{ or } y = 1\} = \mathbb{T}^{p-1} \times \{0,1\},$$

with $\overline{\Lambda}$ the closure of Λ. We partition Γ into two parts representing the lower and upper boundaries of the cylinder:

$$\Gamma^{\text{low}} := \mathbb{T}^{p-1} \times \{0\} \qquad \text{and} \qquad \Gamma^{\text{up}} := \mathbb{T}^{p-1} \times \{1\}.$$

At the microscopic level, given an integer $N \geq 2$, we define Λ_N and Γ_N as the corresponding discrete microscopic sets. Specifically, by letting $\mathbb{T}_N := \mathbb{Z}/N\mathbb{Z}$ the discrete one-dimensional torus of length N, and using the notation $[\![a; b]\!] := [a, b] \cap \mathbb{Z}$ for any $a, b \in \mathbb{R}$,

$$\Lambda_N := \mathbb{T}_N^{p-1} \times [\![1; N-1]\!]$$

represents the cylinder in $\mathbb{Z}^p$ of height $N - 1$ and basis $\mathbb{T}_N^{p-1}$, its boundary being

$$\Gamma_N := \{(i, j) \in \Lambda_N \mid j = 1 \text{ or } N - 1\} = \mathbb{T}_N^{p-1} \times \{1, N-1\}.$$

Similarly, $\Gamma_N = \Gamma_N^{\text{low}} \cup \Gamma_N^{\text{up}}$ with

$$\Gamma_N^{\text{low}} := \mathbb{T}_N^{p-1} \times \{1\} \qquad \text{and} \qquad \Gamma_N^{\text{up}} := \mathbb{T}_N^{p-1} \times \{N-1\}.$$

The elements of $\overline{\Lambda}$ are represented by

$$\hat{x} = (x, y) \qquad \text{and} \qquad \hat{z} = (z, \omega),$$

with $x, z \in \mathbb{T}^{p-1}$ and $y, \omega \in [0, 1]$, while those of Λ_N are symbolized by the letters

$$\hat{\imath} = (i, j) \qquad \text{and} \qquad \hat{\kappa} = (k, \ell),$$

with $i, k \in \mathbb{T}_N^{p-1}$ and $j, \ell \in [\![1; N-1]\!]$.

2.2 Description of the microscopic model

We consider the evolution of two kinds of interacting particles on the lattices Λ_N (the microscopic field) and Γ_N^{low} (the microscopic road). The associated stochastic dynamics is described by the temporal evolution of a Markov process denoted by $(\eta_t, \xi_t)_{t \in [0,T]}$, where $T > 0$ is a given temporal horizon. Particles tied to the dynamics of η (the "field-particles") evolve in the whole microscopic field Λ_N, while the particles corresponding to the dynamics of ξ (the "road-particles") evolve solely on the microscopic road Γ_N^{low}, that stands as the lower frontier of the microscopic field[1]. Both types of particles follow an exclusion rule in its respective environment: each site $\hat{\imath} = (i, j) \in \Lambda_N$ can host at most one field-particle, and similarly, each site $i \in \Gamma_N^{\text{low}}$ can host at most one road-particle. Note in particular that at a site $i \in \Gamma_N^{\text{low}}$, there may be a field-particle *and* a road-particle.

The overall dynamics emerge from the superposition of several independent ones, which are individually specified below and collectively depicted in Figure I:

---

[1]It is important to note that the microscopic road Γ_N^{low} is actually embedded in $\mathbb{Z}^{p-1}$. For the purpose of simplifying notations and facilitating understanding, we often make an identification between $(i, 1)$ and i, establishing a one-to-one correspondence between the lower boundary of the microscopic field and the $(p-1)$-dimensional torus. In line with this simplification, we will use $\xi(i)$ and $\eta(i)$ to denote what are, in fact, $\xi(i, 1)$ and $\eta(i, 1)$. This is a deliberate choice to streamline the expressions without compromising the accuracy of the mathematical representations involved. Similarly, on the upper microscopic boundary Γ_N^{up}, we may write $\eta(i)$ to represent $\eta(i, N-1)$, provided this does not lead to any ambiguity.

- *Diffusion in the field.* Within Λ_N, the field-particles follow a symmetric simple exclusion process and jump at exponential times. The dynamics of this process is as follows: a particle located at site $\hat{\imath}$ awaits an exponential time after which it jumps to a neighboring site $\hat{\kappa}$ with speeded rate $N^2 d$, for some fixed $d > 0$. However, if the site $\hat{\kappa}$ is already occupied, the jump is prevented in accordance with the exclusion rule.

- *Diffusion on the road.* Similarly, the road-particles follow a symmetric simple exclusion process on $\mathbb{T}_N^{p-1}$: a particle positioned at site i awaits an exponential time after which it jumps to a neighboring site k with speeded rate $N^2 D$, for some fixed $D > 0$. However, if the site k is already occupied, the jump is inhibited.

- *Reservoir at the upper field's boundary.* The dynamics defined on the upper boundary Γ_N^{up} act as reservoirs for the field-particles that are much slower compared to the rate of jumps in the bulk. Fix a constant $0 < b < 1$, for each site $\hat{\imath} \in \Gamma_N^{\mathrm{up}}$, the following events occur, according to exponential times that are independent of all others:

  - In the absence of a particle, a new one is generated with rate b.

  - Conversely, if a particle is present, it is eliminated with rate $1 - b$.

- *Exchange dynamics between the lower field's boundary and the road.* We now describe the interacting behavior between the road-particles and the field-particles at the lower boundary of the microscopic field Γ_N^{low}. Fix $\alpha > 0$, for each site $\hat{\imath} = (i, 1) \in \Gamma_N^{\mathrm{low}}$, according to exponential times, the following scenarios may occur:

  - If a road-particle is present and no field-particle exists, then a field-particle is generated at site $\hat{\imath}$ with speeded rate $N\alpha$ ($\heartsuit$). Independently, the road-particle is eliminated with rate α ($\clubsuit$).

  - Conversely, if a field-particle is present without a road-particle, then the field-particle is eliminated with speeded rate $N\alpha$ ($\diamondsuit$). Independently, a road-particle is generated with rate α ($\spadesuit$).

The configuration space is given by
$$S_N := \underbrace{\{0, 1\}^{\Lambda_N}}_{=: S_N^{\mathrm{field}}} \times \underbrace{\{0, 1\}^{\Gamma_N^{\mathrm{low}}}}_{=: S_N^{\mathrm{road}}}$$
which we endow with the product topology. The elements of S_N, referred to as configurations, are denoted by (η, ξ). The first marginal η represents a configuration within the state space S_N^{field}. To be more specific, in a given configuration η, for any $\hat{\imath}$ in Λ_N, $\eta(\hat{\imath}) = 1$ means the site $\hat{\imath}$ is occupied. Conversely, $\eta(\hat{\imath}) = 0$ signifies that the site $\hat{\imath}$ is empty. Similarly, the second marginal ξ stands for a configuration within the state space S_N^{road}, and for any i in Γ_N^{low}, the value $\xi(i)$ indicates the occupancy status of particle at site i in a given configuration ξ.

For any configuration η in S_N^{field} (resp. ξ in S_N^{road}), and any sites $\hat{\imath}, \hat{\kappa}$ in Λ_N (resp. i, k in Γ_N^{low}), let $\eta^{\hat{\imath}, \hat{\kappa}}$ (resp. $\xi^{i,k}$) be the configuration obtained from η (resp. ξ) by switching the values at $\hat{\imath}$ and $\hat{\kappa}$ (resp. i and k), namely

$$(\eta^{\hat{\imath},\hat{\kappa}})(\hat{m}) = \begin{cases} \eta(\hat{\imath}) & \text{if } \hat{m} = \hat{\kappa}, \\ \eta(\hat{\kappa}) & \text{if } \hat{m} = \hat{\imath}, \\ \eta(\hat{m}) & \text{otherwise}, \end{cases} \qquad \left(\text{resp.} \quad (\xi^{i,k})(m) = \begin{cases} \xi(i) & \text{if } m = k, \\ \xi(k) & \text{if } m = i, \\ \xi(m) & \text{otherwise} \end{cases} \right).$$

For the sites $\hat{\imath} \in \Gamma_N$ (resp. $i \in \Gamma_N^{\text{low}}$), let $\eta^{\hat{\imath}}$ (resp. ξ^i) be the configuration obtained from η (resp. ξ) by flipping the occupation number at site $\hat{\imath}$ (resp. i), namely

$$(\eta^{\hat{\imath}})(\hat{m}) = \begin{cases} 1 - \eta(\hat{m}) & \text{if } \hat{m} = \hat{\imath}, \\ \eta(\hat{m}) & \text{otherwise,} \end{cases} \qquad \left(\text{resp.} \quad (\xi^i)(m) = \begin{cases} 1 - \xi(m) & \text{if } (m) = i, \\ \xi(m) & \text{otherwise} \end{cases} \right).$$
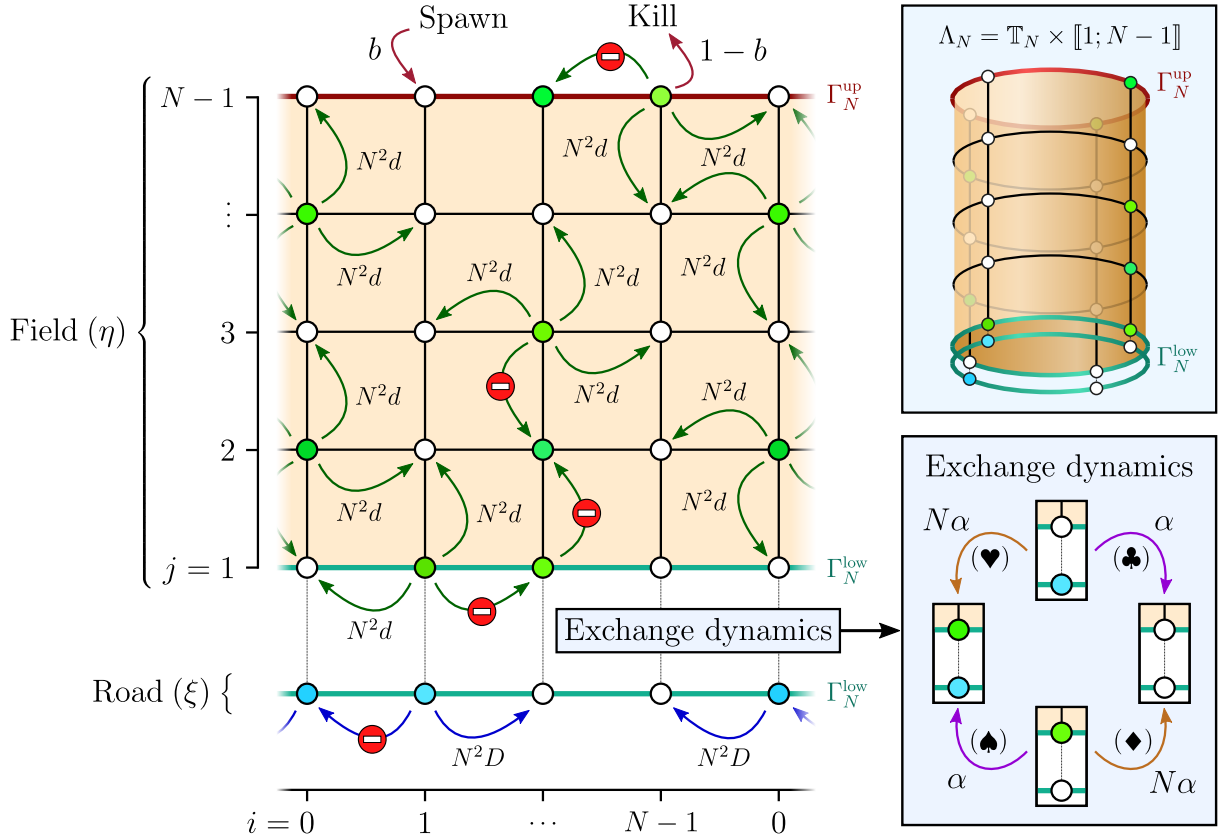

Figure I *— **The microscopic dynamics in dimension** $\mathbf{p} = 2$. In the field, particles are represented by green dots ⬤ and jump towards one of their adjacent sites at exponential times with mean frequency $N^2 d$ — "with rate $N^2 d$" for short. Moves to already occupied sites are prohibited by the exclusion rule, and are indicated by the symbol ⊖. Similarly, particles on the road are depicted by the blue dots ⬤ and jump to their neighboring unoccupied sites with rate $N^2 D$. At the upper boundary of the field Γ_N^{up}, particle emerge in empty sites with rate b and are removed with rate $1 - b$. The interactions at the lower boundary of the field Γ_N^{low} allow the coupling between the field and the road and play a central role in the model. These are detailed in the "Exchange dynamics" panel on the right-hand-side of the figure. For clarity and to facilitate understanding, Γ_N^{low} is represented twice to distinguish between the particles at the lower boundary of the field and those on the road. Notice also that not all possible jumps are represented.*

Fix $\alpha, d, D \in (0, \infty)$, and $0 < b < 1$. The generator of the microscopic dynamics, $\mathcal{L}_N : \mathbb{R}^{S_N} \to \mathbb{R}^{S_N}$, is split as follows:

$$\mathcal{L}_N = \underbrace{N^2 \, \mathbb{I}_N^{\text{field}}}_{\text{field diffusion}} + \overbrace{N^2 \, \mathbb{I}_N^{\text{road}}}^{\text{road diffusion}} + \underbrace{N \, L_N^{\text{Rob}}}_{\substack{\text{lower Robin} \\ \text{condition}}} + \overbrace{L_N^{\text{reac}}}^{\substack{\text{road} \\ \text{reaction}}} + \underbrace{L_N^{\text{up}}}_{\substack{\text{upper} \\ \text{reservoir}}}, \tag{2.1}$$

where, for any $f : S_N \to \mathbb{R}$ and any $(\eta, \xi) \in S_N$,

$$\left(\mathbb{L}_N^{\text{field}} f\right)(\eta, \xi) = \frac{d}{2} \sum_{\substack{\hat{\imath}, \hat{k} \in \Lambda_N \\ |\hat{\imath} - \hat{k}| = 1}} \left[f(\eta^{\hat{\imath}, \hat{k}}, \xi) - f(\eta, \xi) \right], \tag{2.2}$$

$$\left(\mathbb{L}_N^{\text{road}} f\right)(\eta, \xi) = \frac{D}{2} \sum_{\substack{i, k \in \Gamma_N^{\text{low}} \\ |i - k| = 1}} \left[f(\eta, \xi^{i,k}) - f(\eta, \xi) \right], \tag{2.3}$$

$$\left(L_N^{\text{Rob}} f\right)(\eta, \xi) = \alpha \sum_{i \in \Gamma_N^{\text{low}}} \left(\eta(i) - \xi(i) \right)^2 \left[f(\eta^i, \xi) - f(\eta, \xi) \right], \tag{2.4}$$

$$\left(L_N^{\text{reac}} f\right)(\eta, \xi) = \alpha \sum_{i \in \Gamma_N^{\text{low}}} \left(\eta(i) - \xi(i) \right)^2 \left[f(\eta, \xi^i) - f(\eta, \xi) \right], \tag{2.5}$$

$$\left(L_N^{\text{up}} f\right)(\eta, \xi) = \sum_{i \in \Gamma_N^{\text{up}}} \left(b(1 - \eta(i)) + (1 - b)\eta(i) \right) \left[f(\eta^i, \xi) - f(\eta, \xi) \right]. \tag{2.6}$$

In (2.2) and (2.3), we use $|\cdot|$ to denote the infinity norm in $\mathbb{R}^p$, that is

$$\forall \hat{\imath} \in \Lambda_N, \qquad |\hat{\imath}| = \max(|i_1|, \cdots, |i_{p-1}|, |j|).$$

Also, we highlight that the flip rate $\alpha(\eta(i) - \xi(i))^2$ in (2.4) and (2.5) arises from equality

$$(1 - \eta(i))\, \xi(i) + \eta(i)\, (1 - \xi(i)) = (\eta(i) - \xi(i))^2,$$

which holds since both $\eta(i)$ and $\xi(i)$ belong to $\{0, 1\}$.

The parts (2.2), (2.3) and (2.6) are rather classical, see [34], [4], [42] for instance. On the other hand, the parts (2.4) and (2.5) are original, their role being to catch the exchange condition in the field-road model. We refer to [28] and [30] for related issues.

For a given time horizon $T > 0$, we denote $(\eta_t, \xi_t)_{t \in [0,T]}$ the Markov process with state space S_N associated to the generator $\mathcal{L}_N$. We define $D([0, T] ; S_N)$ as the path space for càdlàg time trajectories valued in S_N. Given a measure μ_N on S_N, we denote by $\mathbb{P}_N^{\mu_N}$ the probability measure on $D([0, T] ; S_N)$ induced by μ_N and $(\eta_t, \xi_t)_{t \in [0,T]}$, and we write $\mathbb{E}_N^{\mu_N}$ the expectation with respect to $\mathbb{P}_N^{\mu_N}$. Moreover, the notation $\langle \cdot, \cdot \rangle_{\mu_N}$ refers to the scalar product on $L_{\mu_N}^2(S_N)$.

2.3 Functional spaces and macroscopic equations

For any integers n and m, we define the functional spaces

$$\mathcal{C}^{n,m}([0, T] \times \overline{\Lambda}) \qquad \text{and} \qquad \mathcal{C}^{n,m}([0, T] \times \mathbb{T}^{p-1})$$

which respectively consist of functions

$$G = G(t, \hat{x}) : [0, T] \times \overline{\Lambda} \to \mathbb{R} \qquad \text{and} \qquad H = H(t, x) : [0, T] \times \mathbb{T}^{p-1} \to \mathbb{R},$$

that possess n continuous derivatives with respect to the time variable on $[0, T]$, and m continuous derivatives with respect to the spatial variable on $\overline{\Lambda}$ and $\mathbb{T}^{p-1}$ respectively. We also introduce the subset $\mathcal{C}_c^{n,m}([0, T] \times \overline{\Lambda})$ of functions with compact support in $[0, T] \times \Lambda$

within $\mathcal{C}^{n,m}([0,T] \times \overline{\Lambda})$. Similarly, we denote $\mathcal{C}^m(\overline{\Lambda})$ and $\mathcal{C}^m(\mathbb{T}^{p-1})$ the sets of functions with m continuous derivatives on $\overline{\Lambda}$ and $\mathbb{T}^{p-1}$ respectively.

In the whole document, if φ is a function that depends both on the time and the spatial variables, the abbreviation $\varphi(t)$ naturally stands for $\varphi(t, \cdot)$.

In the sequel, the notations $\langle \cdot, \cdot \rangle_\Lambda$ and $\langle \cdot, \cdot \rangle_{\mathbb{T}^{p-1}}$ respectively represent the $L^2(\Lambda)$ and $L^2(\mathbb{T}^{p-1})$ inner products.

We consider the Sobolev space $\mathcal{H}^1(\Lambda)$ as the set of functions g in $L^2(\Lambda)$ such that for every q ranging from 1 to p, there is an element $\partial_{e_q} g$ in $L^2(\Lambda)$, for which we have

$$\forall G \in \mathcal{C}_c^\infty(\Lambda), \qquad \langle \partial_{e_q} G, g \rangle_\Lambda = -\langle G, \partial_{e_q} g \rangle_\Lambda,$$

where $\partial_{e_q} \varphi$ denotes the derivative with respect to the qth canonical vector of $\mathbb{R}^p$. We then define the norms on the Sobolev space $\mathcal{H}^1(\Lambda)$ by

$$\|g\|_{\mathcal{H}^1(\Lambda)} := \left(\|g\|_{L^2(\Lambda)}^2 + \sum_{q=1}^p \|\partial_{e_q} g\|_{L^2(\Lambda)}^2 \right)^{1/2}.$$

With respect to a given Banach space B (most of the time, $B = \mathcal{H}^1(\Lambda)$ or $B = L^2(\mathbb{T}^{p-1})$), we define $L^2(0, T; B)$ as the function space composed of maps $\varphi : [0, T] \to B$ satisfying

$$\int_0^T \|\varphi(t)\|_B^2 \, dt < +\infty.$$

In order to define the value of an element g in $\mathcal{H}^1(\Lambda)$ at the boundaries $\Gamma = \Gamma^{\text{up}} \cup \Gamma^{\text{low}}$, we need to introduce the notion of trace. The trace operator on the space $\mathcal{H}^1(\Lambda)$ can be defined as the bounded linear operator

$$\text{Tr} : \mathcal{H}^1(\Lambda) \to L^2(\Gamma)$$

such that Tr extends the classical trace, that is

$$\forall g \in \mathcal{H}^1(\Lambda) \cap \mathcal{C}^0(\overline{\Lambda}), \qquad \text{Tr}(g) = g|_\Gamma.$$

We refer to [24, Part II, Section 5] for a detailed survey of the trace operator. In the sequel, for any point $\hat{x} = (x, y)$ in Γ and any function g belonging to the space $L^2(0, T; \mathcal{H}^1(\Lambda))$, all the expressions $g(t, \hat{x})$, $g(t, x)$ or $g(t, \cdot)|_\Gamma(x)$ represent the trace operator applied to $g(t, \cdot)$ at position $\hat{x} \in \Gamma$. Additionally, observe that $\partial_y g(t, \cdot)$ (resp. $-\partial_y g(t, \cdot)$) stands for the outer normal derivative of the function $g(t, \cdot)$ on the boundary Γ^{up} (resp. Γ^{low}).

We are now in a position to introduce our notion of weak solution for the field-road problem (1.2)-(1.3).

Definition 2.1 (Solving the field-road system) *Fix a time horizon $T > 0$. For any measurable initial data $v_0 : \Lambda \to [0, 1]$ and $u_0 : \mathbb{T}^{p-1} \to [0, 1]$, a couple of functions (v, u) is said to be a weak solution to the initial value problem (1.2)-(1.3) as soon as the following two conditions (W1)-(W2) hold true:*

(W1) *$v \in L^2(0, T; \mathcal{H}^1(\Lambda))$ and $u \in L^2(0, T; L^2(\mathbb{T}^{p-1}))$.*

(W2) *For any $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$, any $H \in \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^{p-1})$, there holds*

$$\langle v(T), G(T) \rangle_\Lambda - \langle v_0, G(0) \rangle_\Lambda = \int_0^T \langle v(s), \partial_s G(s) \rangle_\Lambda \, ds + \int_0^T \langle v(s), d\Delta G(s) \rangle_\Lambda \, ds$$

$$- \int_0^T \langle v|_{y=1}(s), d\partial_y G|_{y=1}(s) \rangle_{\mathbb{T}^{p-1}} \, ds + \int_0^T \langle v|_{y=0}(s), d\partial_y G|_{y=0}(s) \rangle_{\mathbb{T}^{p-1}} \, ds \qquad (2.7)$$

$$+ \int_0^T \alpha \, \langle u(s) - v|_{y=0}(s), G|_{y=0}(s) \rangle_{\mathbb{T}^{p-1}} \, ds,$$

together with

$$\langle u(T), H(T) \rangle_{\mathbb{T}^{p-1}} - \langle u_0, H(0) \rangle_{\mathbb{T}^{p-1}} = \int_0^T \langle u(s), \partial_s H(s) \rangle_{\mathbb{T}^{p-1}} + \int_0^T \langle u(s), D\Delta_x H(s) \rangle_{\mathbb{T}^{p-1}} \, ds$$

$$+ \int_0^T \alpha \, \langle v|_{y=0}(s) - u(s), H(s) \rangle_{\mathbb{T}^{p-1}} \, ds. \qquad (2.8)$$

2.4 Main Results

We use the notations $\mathcal{M}^{\text{field}}$ and $\mathcal{M}^{\text{road}}$ to denote the sets of positive Radon measures on Λ and Γ^{low} whose total mass is bounded by 1, and we define the space $\mathcal{M}$ as the Cartesian product $\mathcal{M}^{\text{field}} \times \mathcal{M}^{\text{road}}$. We denote the integrals of functions against measures indifferently for Λ or Γ^{low}, namely, for any measures $\mu \in \mathcal{M}^{\text{field}}$ and $\nu \in \mathcal{M}^{\text{road}}$, and any functions $G \in L^1_\mu(\Lambda)$ and $H \in L^1_\mu(\mathbb{T}^{p-1})$ by

$$\langle \mu, G \rangle = \int_\Lambda G(\hat{x}) \, \mu(d\hat{x}) \qquad \text{and} \qquad \langle \mu, H \rangle = \int_{\mathbb{T}^{p-1}} H(x) \, \nu(dx).$$

We endow $\mathcal{M}$, $\mathcal{M}^{\text{field}}$ and $\mathcal{M}^{\text{road}}$ with a topology induced by the weak convergence of measures. It is worth mentioning that all these spaces are compact and Polish.

The empirical measure of a configuration $(\eta, \xi) \in S_N$ is defined as $\pi_N(\eta, \xi)$, where the map $\pi_N : S_N \to \mathcal{M}$ is given by

$$\pi_N(\eta, \xi) := \left(\underbrace{\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} \eta(\hat{\imath}) \delta_{\hat{\imath}/N}}_{=: \, \pi_N^{\text{field}}(\eta)}, \quad \underbrace{\frac{1}{N^{p-1}} \sum_{i \in \Gamma_N^{\text{low}}} \xi(i) \delta_{i/N}}_{=: \, \pi_N^{\text{road}}(\xi)} \right). \qquad (2.9)$$

In (2.9), the notation $\delta_{\hat{\imath}/N}$ (resp. $\delta_{i/N}$) stands for the Dirac mass at $\hat{\imath}/N$ (resp. i/N). For any configuration $(\eta, \xi) \in S_N$, any $G \in \mathcal{C}^0(\overline{\Lambda})$ and any $H \in \mathcal{C}^0(\mathbb{T}^{p-1})$, we denote

$$\langle \pi_N(\eta, \xi), [G, H] \rangle := \langle \pi_N^{\text{field}}(\eta), G \rangle + \langle \pi_N^{\text{road}}(\xi), H \rangle.$$

We introduce $(\pi_N(t))_{t \in [0,T]} := (\pi_N(\eta_t, \xi_t))_{t \in [0,T]}$, the Markov process on the state space $\mathcal{M}$ induced from $(\eta_t, \xi_t)_{t \in [0,T]}$. The trajectories of this process occupy $D([0,T]; \mathcal{M})$, the designated path space for càdlàg time trajectories valued in $\mathcal{M}$. We endow $D([0,T]; \mathcal{M})$ with the Skorokhod topology. For further details regarding this topology, we refer the reader to [14], which provides an extensive survey on this subject. For $G \in \mathcal{C}^{0,0}([0,T] \times \overline{\Lambda})$, $H \in \mathcal{C}^{0,0}([0,T] \times \mathbb{T}^{p-1})$, and $t \in [0,T]$, we denote

$$\langle \pi_N(t), [G(t), H(t)] \rangle := \langle \pi_N^{\text{field}}(t), G(t) \rangle + \langle \pi_N^{\text{road}}(t), H(t) \rangle,$$

where $\pi_N^{\text{field}}(t) := \pi_N^{\text{field}}(\eta_t)$ and $\pi_N^{\text{road}}(t) := \pi_N^{\text{road}}(\xi_t)$.

Given an initial probability measure μ_N on S_N, we define, for $N \geq 2$, the probability measure $\mathbb{Q}_N^{\mu_N} := \mathbb{P}_N^{\mu_N}(\pi_N^{-1})$ on the set of measures $\mathcal{M}$, as the law of the Markov process

$$(\pi_N(t))_{t \in [0,T]} = (\pi_N(\eta_t, \xi_t))_{t \in [0,T]}.$$

Essentially, $\mathbb{Q}_N^{\mu_N}$ allows to provide a description through measures on the macroscopic space of the state distribution of the process $(\eta_t, \xi_t)_{t \in [0,T]}$ when initiated from the measure μ_N.

Definition 2.2 (Sequence of measures associated with the initial data) *Let*

$$v_0 : \Lambda \to [0,1] \quad and \quad u_0 : \mathbb{T}^{p-1} \to [0,1]$$

be two measurable functions. We say that a sequence of probability measures

$$(\mu_N)_{N \geq 2} = (\mu_N^{\text{field}}, \mu_N^{\text{road}})_{N \geq 2}$$

on $S_N = S_N^{\text{field}} \times S_N^{\text{road}}$ is associated with (v_0, u_0) if

$$\lim_{N \to \infty} \mu_N^{\text{field}}\left[\eta \in S_N^{\text{field}} : \left| \langle \pi_N^{\text{field}}(\eta), G \rangle - \langle v_0, G \rangle_\Lambda \right| \geq \delta \right] = 0, \tag{2.10}$$

and

$$\lim_{N \to \infty} \mu_N^{\text{road}}\left[\xi \in S_N^{\text{road}} : \left| \langle \pi_N^{\text{road}}(\xi), H \rangle - \langle u_0, H \rangle_{\Gamma^{\text{low}}} \right| \geq \delta \right] = 0, \tag{2.11}$$

for any $\delta > 0$, any $G \in \mathcal{C}^0(\overline{\Lambda})$ and $H \in \mathcal{C}^0(\mathbb{T}^{p-1})$.

Remark 2.3 *Observe in Definition 2.2 that we ask v_0 and u_0 to be valued in $[0,1]$. This condition cannot be relaxed because of the exclusion rule, that enforces the sites to host at most one particle. To be convinced with this, consider the case $u_0 \equiv 2$, and take $H \equiv 1$. We then have $\langle u_0, H \rangle_{\Gamma^{\text{low}}} = 2$, and*

$$\langle \pi_N^{\text{road}}(\xi), H \rangle = \frac{1}{N^{p-1}} \sum_{i \in \Gamma_N^{\text{low}}} \xi(i) \leq \frac{1}{N^{p-1}} \sum_{i \in \Gamma_N^{\text{low}}} 1 = 1,$$

so that no configuration allows the limit (2.11) to hold, and the data $u_0 \equiv 2$ is unreachable.

Here is the main contribution of the present work.

Theorem 2.4 (Hydrodynamic limit) *Fix a time horizon $T > 0$. Let $v_0 : \Lambda \to [0,1]$ and $u_0 : \mathbb{T}^{p-1} \to [0,1]$ be two measurable functions, and $(\mu_N)_{N \geq 2}$ a sequence of initial probability measures on S_N associated with (v_0, u_0) in the sense of Definition 2.2. Then the sequence of probability measures $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$ converges weakly towards some $\mathbb{Q}_\infty$ which is concentrated on the path*

$$\left(\pi^{\text{field}}(t, dx\, dy),\ \pi^{\text{road}}(t, dx) \right)_{t \in [0,T]} = \left(v(t, x, y)dx\, dy,\ u(t, x)dx \right)_{t \in [0,T]},$$

where (v, u) is the unique weak solution to the Cauchy problem (1.2)-(1.3) in the sense of Definition 2.1. In particular, for any $t \in [0,T]$, any $\delta > 0$, and any test function

$$G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda}) \quad and \quad H \in \mathcal{C}^{1,2}([0,T] \times \Gamma^{\text{low}}),$$

we have

$$\lim_{N \to \infty} \mathbb{P}_N^{\mu_N}\left[\eta_t \in S_N^{\text{field}} : \left| \langle \pi_N^{\text{field}}(t), G(t) \rangle - \langle v(t), G(t) \rangle_\Lambda \right| > \delta \right] = 0,$$

and

$$\lim_{N \to \infty} \mathbb{P}_N^{\mu_N}\left[\xi_t \in S_N^{\text{road}} : \left| \langle \pi_N^{\text{road}}(t), H(t) \rangle - \langle u(t), H(t) \rangle_{\Gamma^{\text{low}}} \right| > \delta \right] = 0.$$

2.5 The steps to prove Theorem 2.4 and organisation of the paper

The proof of the hydrodynamic limit, as outlined in Theorem 2.4, is inspired by works such as those by Kipnis and Landim [34] or Baldasso *et al.* [4], and unfolds in three distinct steps. The first one is to prove the tightness of the probability measures $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$ within the Skorokhod topology. This point, established in Section 3, is crucial to ensure the existence of accumulation points for this sequence. Following this, we characterize in Section 4 the limit points $\mathbb{Q}_\infty$ of the sequence $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$. Specifically, it is demonstrated that every $\mathbb{Q}_\infty$ is concentrated on set of measure trajectories that are absolutely continuous with respect to the Lebesgue measure at every moment (subsection 4.1), and that the corresponding densities satisfy conditions (W1)-(W2) (subsections 4.2 and 4.4) that characterize our notion of solution. This proves that the density of the measures loaded by $\mathbb{Q}_\infty$ are weak solutions to the Cauchy problem (1.2)-(1.3). The third and final step, performed in Section 5, consists in showing — thanks to very adequate test functions — that the Cauchy problem (1.2)-(1.3) admits a unique solution.

3 Martingales and tightness

3.1 Martingales

We now proceed to explain the martingales associated to our system. Fix a couple of functions (G, H) with $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$ and $H \in \mathcal{C}^{1,2}([0,T] \times \Gamma^{\mathrm{low}})$, and consider the martingale $\mathscr{M}_N = \mathscr{M}_{N,G,H}$ with respect to the natural filtration $\sigma((\eta_s, \xi_s)_{0 \leq s \leq t})$ given for any $t \in [0,T]$ by

$$\mathscr{M}_N(t) := \langle \pi_N(t), [G(t), H(t)] \rangle - \langle \pi_N(0), [G(0), H(0)] \rangle - \int_0^t (\partial_s + \mathcal{L}_N)\left(\langle \pi_N(s), [G(s), H(s)] \rangle\right) ds. \quad (3.1)$$

Although the family of martingales associated to a Markov process considered in [34, Appendix 1, Section 5] is stated for functions in $\mathcal{C}^{2,2}$, we use functions $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$ and $H \in \mathcal{C}^{1,2}([0,T] \times \Gamma^{\mathrm{low}})$. This is justified because the martingale formulation (3.1), on the bounded interval $[0,T]$, only requires the first partial derivative with respect to time of G and H, and the use of $\mathcal{C}^{1,2}$ functions is sufficient to prove that $\mathscr{M}_{N,G,H}$ is a martingale. Indeed, it suffices to follow the proof of [34, Appendix 1, Section 5, Lemma 5.1], and note that the only step where the second partial derivative with respect to time is needed is to estimate the first term in formula (5.2) of [34] but, in our case, the uniform (since on compacts) continuity of the first partial derivative with respect to time is enough.

Expanding the empirical measure π_N with (2.9) in (3.1), we see that $\mathscr{M}_N$ can be split into $\mathscr{M}_{N,G}^{\mathrm{field}} + \mathscr{M}_{N,H}^{\mathrm{road}}$ where

$$\mathscr{M}_{N,G}^{\mathrm{field}}(t) = \langle \pi_N^{\mathrm{field}}(t), G(t) \rangle - \langle \pi_N^{\mathrm{field}}(0), G(0) \rangle - \int_0^t [\partial_s + \mathcal{L}_N]\left[\langle \pi_N^{\mathrm{field}}(s), G(s) \rangle\right] ds,$$

$$\mathscr{M}_{N,H}^{\mathrm{road}}(t) = \langle \pi_N^{\mathrm{road}}(t), H(t) \rangle - \langle \pi_N^{\mathrm{road}}(0), H(0) \rangle - \int_0^t [\partial_s + \mathcal{L}_N]\left[\langle \pi_N^{\mathrm{road}}(s), H(s) \rangle\right] ds. \quad (3.2)$$

The quadratic variation of $\mathscr{M}_N$ is given by the martingale $\mathscr{N}_N = \mathscr{N}_{N,G,H}$ defined for any $t \in [0,T]$ as

$$\mathscr{N}_N(t) := [\mathscr{M}_N(t)]^2 - \int_0^t \mathscr{B}_N(s) ds, \quad (3.3)$$

where

$$\mathscr{B}_N(s) := \mathcal{L}_N\left[\langle \pi_N(s), [G(s), H(s)]\rangle^2\right] - 2\langle \pi_N(s), [G(s), H(s)]\rangle \times \mathcal{L}_N\left[\langle \pi_N(s), [G(s), H(s)]\rangle\right]. \quad (3.4)$$

A proof that $\mathscr{M}_N$ and $\mathscr{N}_N$ are martingales with respect to the natural filtration is classical and follows the very same lines as [34, Appendix 1, Section 5].

In the integral terms of $\mathscr{M}_{N,G}^{\text{field}}$ and $\mathscr{M}_{N,H}^{\text{road}}$ in (3.2) we expand the expressions of π_N^{field}, π_N^{road} (2.9), and $\mathcal{L}_N$ (2.1), and use at some point two discrete summations by parts. After some tedious but straightforward computations — using facts such as $(\eta(i) - \xi(i))^2(1 - 2\eta(i)) = \xi(i) - \eta(i)$ or $(b(1 - \eta(i)) + (1 - b)\eta(i))(1 - 2\eta(i)) = b - \eta(i)$ — we get

$$
\begin{aligned}
\mathscr{M}_{N,G}^{\text{field}}(t) =\ & \langle \pi_N^{\text{field}}(t), G(t)\rangle - \langle \pi_N^{\text{field}}(0), G(0)\rangle - \int_0^t \langle \pi_N^{\text{field}}(s), \partial_s G(s)\rangle ds \\
& - \frac{1}{N^p}\int_0^t \sum_{\hat{\imath} \in \Lambda_N} d\Delta_x^N G(s, \tfrac{\hat{\imath}}{N}) \times \eta_s(\hat{\imath}) ds - \frac{1}{N^p}\int_0^t \sum_{\hat{\imath} \in \Lambda_N \backslash \Gamma_N} d\partial_{yy}^N G(s, \tfrac{\hat{\imath}}{N}) \times \eta_s(\hat{\imath}) ds \\
& + \frac{1}{N^{p-1}}\int_0^t \sum_{i \in \Gamma_N^{\text{up}}} d\partial_y^N G(s, \tfrac{i}{N}) \times \eta_s(i) ds - \frac{1}{N^{p-1}}\int_0^t \sum_{i \in \Gamma_N^{\text{low}}} d\partial_y^N G(s, \tfrac{i}{N}) \times \eta_s(i) ds \\
& - \frac{\alpha}{N^{p-1}}\int_0^t \sum_{i \in \Gamma_N^{\text{low}}} G(s, \tfrac{i}{N}) \times (\xi_s(i) - \eta_s(i)) ds - \frac{1}{N^p}\int_0^t \sum_{i \in \Gamma_N^{\text{up}}} G(s, \tfrac{i}{N}) \times (b - \eta_s(i)) ds,
\end{aligned}
\quad (3.5)
$$

and

$$
\begin{aligned}
\mathscr{M}_{N,H}^{\text{road}}(t) =\ & \langle \pi_N^{\text{road}}(t), H(t)\rangle - \langle \pi_N^{\text{road}}(0), H(0)\rangle - \int_0^t \langle \pi_N^{\text{road}}(s), \partial_s H(s)\rangle ds \\
& - \frac{1}{N^{p-1}}\int_0^t \sum_{i \in \Gamma_N^{\text{low}}} D\Delta_x^N H(s, \tfrac{i}{N}) \times \xi_s(i) ds - \frac{\alpha}{N^{p-1}}\int_0^t \sum_{i \in \Gamma_N^{\text{low}}} H(s, \tfrac{i}{N}) \times (\eta_s(i) - \xi_s(i)) ds.
\end{aligned}
\quad (3.6)
$$

In (3.5) and (3.6), the discrete Laplacians Δ_x^N and ∂_{yy}^N are defined by (time dependency is locally drop for the sake of clarity)

$$
\begin{aligned}
\forall i \in \mathbb{T}_N^{p-1}, \qquad & \Delta_x^N H(\tfrac{i}{N}) := N^2 \sum_{q=1}^{p-1}\left[H(\tfrac{i+\tilde{e}_q}{N}) - 2H(\tfrac{i}{N}) + H(\tfrac{i-\tilde{e}_q}{N})\right], \\
\forall (i,j) \in \Lambda_N, \qquad & \Delta_x^N G(\tfrac{i}{N}, \tfrac{j}{N}) := N^2 \sum_{q=1}^{p-1}\left[G(\tfrac{i+\tilde{e}_q}{N}, \tfrac{j}{N}) - 2G(\tfrac{i}{N}, \tfrac{j}{N}) + G(\tfrac{i-\tilde{e}_q}{N}, \tfrac{j}{N})\right], \\
\forall (i,j) \in \Lambda_N \backslash \Gamma_N, \qquad & \partial_{yy}^N G(\tfrac{i}{N}, \tfrac{j}{N}) := N^2\left[G(\tfrac{i}{N}, \tfrac{j+1}{N}) - 2G(\tfrac{i}{N}, \tfrac{j}{N}) + G(\tfrac{i}{N}, \tfrac{j-1}{N})\right],
\end{aligned}
\quad (3.7)
$$

where $(\tilde{e}_q)_{1 \le q \le p-1}$ denotes the canonical basis of $\mathbb{R}^{p-1}$, and the discrete derivative ∂_y^N at the boundary of the microscopic field by (time dependency is locally drop for the sake of clarity)

$$
\partial_y^N G(\tfrac{i}{N}, \tfrac{j}{N}) := \begin{cases} N\left[G(\tfrac{i}{N}, \tfrac{j+1}{N}) - G(\tfrac{i}{N}, \tfrac{j}{N})\right] & \text{if } j = 1, \\ N\left[G(\tfrac{i}{N}, \tfrac{j}{N}) - G(\tfrac{i}{N}, \tfrac{j-1}{N})\right] & \text{if } j = N-1. \end{cases}
\quad (3.8)
$$

Similarly, we also develop $\mathscr{B}_N$ from (3.4) with the expression of $\mathcal{L}_N$ in (2.1). The

computations show that $\mathscr{B}_N$ can be split into $\mathscr{B}_{N,G}^{\text{field}} + \mathscr{B}_{N,H}^{\text{road}}$ with

$$
\begin{aligned}
\mathscr{B}_{N,G}^{\text{field}}(s) = {} & \frac{d/2}{N^{2p}} \sum_{\substack{\hat{\imath},\hat{\kappa} \in \Lambda_N \\ |\hat{\imath}-\hat{\kappa}|=1}} \left[\eta_s(\hat{\imath}) - \eta_s(\hat{\kappa}) \right]^2 \times \left[N(G(s,\tfrac{\hat{\imath}}{N}) - G(s,\tfrac{\hat{\kappa}}{N})) \right]^2 \\
& + \frac{\alpha}{N^{2p-1}} \sum_{i \in \Gamma_N^{\text{low}}} \left[\eta_s(i) - \xi_s(i) \right]^2 \times \left[G(s,\tfrac{i}{N}) \right]^2 \\
& + \frac{1}{N^{2p}} \sum_{i \in \Gamma_N^{\text{up}}} \left[b(1-\eta_s(i)) + (1-b)\eta_s(i) \right] \times \left[G(s,\tfrac{i}{N}) \right]^2
\end{aligned}
\tag{3.9}
$$

and

$$
\begin{aligned}
\mathscr{B}_{N,H}^{\text{road}}(s) = {} & \frac{D/2}{N^{2(p-1)}} \sum_{\substack{i,k \in \Gamma_N^{\text{low}} \\ |i-k|=1}} \left[\xi_s(i) - \xi_s(k) \right]^2 \times \left[N(H(s,\tfrac{i}{N}) - H(s,\tfrac{k}{N})) \right]^2 \\
& + \frac{\alpha}{N^{2(p-1)}} \sum_{i \in \Gamma_N^{\text{low}}} \left[\eta_s(i) - \xi_s(i) \right]^2 \times \left[H(s,\tfrac{i}{N}) \right]^2 .
\end{aligned}
\tag{3.10}
$$

3.2 Tightness

We are now in position to prove the tightness of $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$.

Proposition 3.1 (Tightness) *For any sequence of initial measures $(\mu_N)_{N \geq 2}$ on S_N, the sequence $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$ is tight in the Skorokhod topology of $D([0,T]\,;\mathcal{M})$.*

Proof of Proposition 3.1. In accordance with the method presented in [34, Chapter 4, Section 1], it is enough to verify the following two statements $G \in \mathcal{C}^2(\overline{\Lambda})$ and any $H \in \mathcal{C}^2(\Gamma^{\text{low}})$:

(T1) For all $t \in [0,T]$ and all $\varepsilon > 0$, there is $M > 0$ such that

$$
\sup_{N \geq 2} \left\{ \mathbb{Q}_N^{\mu_N}\left(\left| \langle \pi_N(t), [G,H] \rangle \right| \geq M \right) \right\} \leq \varepsilon.
\tag{3.11}
$$

(T2) For all $\varepsilon > 0$, we have

$$
\lim_{\delta \to 0^+} \limsup_{N \to \infty} \left\{ \mathbb{Q}_N^{\mu_N}\left(\sup_{\substack{0 \leq s,t \leq T \\ |s-t| \leq \delta}} \left| \langle \pi_N(t), [G,H] \rangle - \langle \pi_N(s), [G,H] \rangle \right| \geq \varepsilon \right) \right\} = 0.
\tag{3.12}
$$

Note that, within this proof, we require the test functions G and H to depend solely on the spatial variable.

• *Proof of* (T1). Since the empirical measures π_N^{field} and π_N^{road} are both bounded by 1, we almost surely have

$$
\left| \langle \pi_N^{\text{field}}(t), G \rangle \right| \leq \|G\|_{L^\infty(\Lambda)} \qquad \text{and} \qquad \left| \langle \pi_N^{\text{road}}(t), H \rangle \right| \leq \|H\|_{L^\infty(\Gamma^{\text{low}})}.
$$

As a result, (3.11) is achieved with $M = \varepsilon^{-1}(\|G\|_{L^\infty(\Lambda)} + \|H\|_{L^\infty(\Gamma^{\mathrm{low}})})$.

• *Proof of* (T2). By expanding the terms $\langle \pi_N(t), [G, H] \rangle$ and $\langle \pi_N(s), [G, H] \rangle$ with those of the Dynkin's formula in (3.1) and using the Markov and the triangular inequalities, we can see that (3.12) holds if, for any $\varepsilon > 0$, we prove the following limits:

$$\lim_{\delta \to 0^+} \limsup_{N \to \infty} \left\{ \mathbb{P}_N^{\mu_N} \left(\sup_{\substack{0 \le s, t \le T \\ |s-t| \le \delta}} \left| \mathscr{M}_N(t) - \mathscr{M}_N(s) \right| > \varepsilon \right) \right\} = 0, \tag{3.13}$$

$$\lim_{\delta \to 0^+} \limsup_{N \to \infty} \left\{ \mathbb{P}_N^{\mu_N} \left(\sup_{\substack{0 \le s, t \le T \\ |s-t| \le \delta}} \left| \int_s^t \mathcal{L}_N \left[\langle \pi_N^{\mathrm{field}}(r), G \rangle \right] dr \right| > \varepsilon \right) \right\} = 0, \tag{3.14}$$

$$\lim_{\delta \to 0^+} \limsup_{N \to \infty} \left\{ \mathbb{P}_N^{\mu_N} \left(\sup_{\substack{0 \le s, t \le T \\ |s-t| \le \delta}} \left| \int_s^t \mathcal{L}_N \left[\langle \pi_N^{\mathrm{road}}(r), H \rangle \right] dr \right| > \varepsilon \right) \right\} = 0. \tag{3.15}$$

We start to show (3.13). From Doob's inequality, for any $\varepsilon > 0$,

$$\mathbb{P}_N^{\mu_N} \left(\sup_{\substack{0 \le s, t \le T \\ |s-t| \le \delta}} \left| \mathscr{M}_N(t) - \mathscr{M}_N(s) \right| > \varepsilon \right) \le \mathbb{P}_N^{\mu_N} \left(\sup_{0 \le t \le T} \left| \mathscr{M}_N(t) \right| > \varepsilon/2 \right)$$

$$\begin{aligned} &\le \frac{4}{\varepsilon^2} \mathbb{E}_N^{\mu_N} \left(\left[\mathscr{M}_N(T) \right]^2 \right) \\ &\stackrel{(3.3)}{=} \frac{4}{\varepsilon^2} \mathbb{E}_N^{\mu_N} \left(\mathscr{N}_N(T) + \int_0^T \mathscr{B}_N(s) ds \right) \\ &= \frac{4}{\varepsilon^2} \mathbb{E}_N^{\mu_N} \left(\int_0^T \mathscr{B}_N(s) ds \right), \end{aligned} \tag{3.16}$$

where the second line arises from the martingale property. Now observe from (3.9) and (3.10) that, for any $s \in [0, T]$,

$$|\mathscr{B}_{N,G}^{\mathrm{field}}(s)| \le \frac{d/2}{N^{2p}} \|\nabla G\|_{L^\infty(\Lambda)}^2 \left(\sum_{\substack{\hat{\imath}, \hat{k} \in \Lambda_N \\ |\hat{\imath} - \hat{k}| = 1}} 1 \right) + \frac{\alpha}{N^{2p-1}} \|G\|_{L^\infty(\Lambda)}^2 \left(\sum_{i \in \Gamma_N^{\mathrm{low}}} 1 \right) + \frac{1}{N^{2p}} \|G\|_{L^\infty(\Lambda)}^2 \left(\sum_{i \in \Gamma_N^{\mathrm{low}}} 1 \right)$$

and

$$|\mathscr{B}_{N,H}^{\mathrm{road}}(s)| \le \frac{D/2}{N^{2(p-1)}} \|\nabla_x H\|_{L^\infty(\mathbb{T}^{p-1})}^2 \left(\sum_{\substack{i, k \in \Gamma_N^{\mathrm{low}} \\ |i-k| = 1}} 1 \right) + \frac{\alpha}{N^{2(p-1)}} \|H\|_{L^\infty(\mathbb{T}^{p-1})}^2 \left(\sum_{i \in \Gamma_N^{\mathrm{low}}} 1 \right).$$

As a result, we have $|\mathscr{B}_{N,G}^{\mathrm{field}}(s)| = O(1/N^p)$ and $|\mathscr{B}_{N,H}^{\mathrm{road}}(s)| = O(1/N^{p-1})$, and then $|\mathscr{B}_N(s)| = O(1/N^{p-1})$. Combining those upper bounds with (3.16), the proof of the limit (3.13) is then completed.

We move now on the proofs of (3.14) and (3.15). By similar computations as those used to develop $\mathscr{M}_{N,G}^{\mathrm{field}}$ and $\mathscr{M}_{N,H}^{\mathrm{road}}$ in (3.5) and (3.6), we express the terms under the integrals in (3.14) and (3.15):

$$\begin{aligned} \mathcal{L}_N \left[\langle \pi_N^{\mathrm{field}}(s), G \rangle \right] = &\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} d \Delta_x^N G(\tfrac{\hat{\imath}}{N}) \times \eta_s(\hat{\imath}) + \frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N \setminus \Gamma_N} d \partial_{yy}^N G(\tfrac{\hat{\imath}}{N}) \times \eta_s(\hat{\imath}) \\ &- \frac{1}{N^{p-1}} \sum_{i \in \Gamma_N^{\mathrm{up}}} d \partial_y^N G(\tfrac{i}{N}) \times \eta_s(i) + \frac{1}{N^{p-1}} \sum_{i \in \Gamma_N^{\mathrm{low}}} d \partial_y^N G(\tfrac{i}{N}) \times \eta_s(i) \\ &+ \frac{\alpha}{N^{p-1}} \sum_{i \in \Gamma_N^{\mathrm{low}}} G(\tfrac{i}{N}) \times (\xi_s(i) - \eta_s(i)) + \frac{1}{N^p} \sum_{i \in \Gamma_N^{\mathrm{up}}} G(\tfrac{i}{N}) \times (b - \eta_s(i)) \end{aligned} \tag{3.17}$$

and

$$\mathcal{L}_N\left[\langle\pi_N^{\mathrm{road}}(s),H\rangle\right] = \frac{1}{N^{p-1}}\sum_{i\in\Gamma_N^{\mathrm{low}}}D\Delta_x^N H(\tfrac{i}{N})\times\xi_s(i) + \frac{\alpha}{N^{p-1}}\sum_{i\in\Gamma_N^{\mathrm{low}}}H(\tfrac{i}{N})\times(\eta_s(i)-\xi_s(i)), \quad (3.18)$$

where we recall that the discrete operators Δ_x^N, ∂_{yy}^N, and ∂_y^N are defined in (3.7) and (3.8). Using then some Taylor expansions to control the discrete derivatives in (3.17) and (3.18), we reach

$$\left|\mathcal{L}_N\left[\langle\pi_N^{\mathrm{field}}(s),G\rangle\right]\right| \leq \frac{d}{N^p}\left(\sup_\Lambda|\Delta_x G|+O(1/N)\right)\left(\sum_{\hat{i}\in\Lambda_N}1\right) + \frac{d}{N^p}\left(\sup_\Lambda|\partial_{yy}G|+O(1/N)\right)\left(\sum_{\hat{i}\in\Lambda_N\backslash\Gamma_N}1\right)$$

$$+ \frac{d}{N^{p-1}}\left(\sup_{\Gamma^{\mathrm{up}}}|\partial_y G|+O(1/N)\right)\left(\sum_{i\in\Gamma_N^{\mathrm{up}}}1\right) + \frac{d}{N^{p-1}}\left(\sup_{\Gamma^{\mathrm{low}}}|\partial_y G|+O(1/N)\right)\left(\sum_{i\in\Gamma_N^{\mathrm{low}}}1\right)$$

$$+ \frac{\alpha}{N^{p-1}}\left(\sup_{\Gamma^{\mathrm{low}}}|G|\right)\left(\sum_{i\in\Gamma_N^{\mathrm{low}}}1\right) + \left(\sup_{\Gamma^{\mathrm{up}}}|G|\right)\left(\sum_{i\in\Gamma_N^{\mathrm{up}}}1\right)$$

and

$$\left|\mathcal{L}_N\left[\langle\pi_N^{\mathrm{road}}(s),H\rangle\right]\right| \leq \frac{D}{N^{p-1}}\left(\sup_{\mathbb{T}^{p-1}}|\Delta_x H|+O(1/N)\right)\left(\sum_{i\in\Gamma_N^{\mathrm{low}}}1\right) + \frac{\alpha}{N^{p-1}}\left(\sup_{\mathbb{T}^{p-1}}|H|\right)\left(\sum_{i\in\Gamma_N^{\mathrm{low}}}1\right).$$

From this, it follows that $\left|\mathcal{L}_N\left[\langle\pi_N^{\mathrm{field}}(s),G\rangle\right]\right|$ and $\left|\mathcal{L}_N\left[\langle\pi_N^{\mathrm{road}}(s),H\rangle\right]\right|$ remain bounded. Therefore, for some constant $C>0$, we have

$$\mathbb{P}_N^{\mu_N}\left(\sup_{\substack{0\leq s,t\leq T\\|s-t|\leq\delta}}\left|\int_s^t\mathcal{L}_N\left[\langle\pi_N^{\mathrm{field}}(r),G\rangle\right]dr > \varepsilon\right|\right) \leq \frac{C\delta}{\varepsilon},$$

and

$$\mathbb{P}_N^{\mu_N}\left(\sup_{\substack{0\leq s,t\leq T\\|s-t|\leq\delta}}\left|\int_s^t\mathcal{L}_N\left[\langle\pi_N^{\mathrm{road}}(r),G\rangle\right]dr > \varepsilon\right|\right) \leq \frac{C\delta}{\varepsilon},$$

and the limits (3.14) and (3.15) are a straight consequence. Having verified both conditions (T1) and (T2), we can conclude that the sequence $(\mathbb{Q}_N^{\mu_N})_{N\geq2}$ is tight with respect to the Skorokhod topology on $D([0,T];\mathcal{M})$. This completes the proof. $\qquad\square$

4 Characterization of the limit points of $(\mathbb{Q}_N^{\mu_N})_{N\geq2}$

4.1 The limit points of $(\mathbb{Q}_N^{\mu_N})_{N\geq2}$ load paths with density

Proposition 4.1 (Loading Lebesgue continuous measure processes) *Any limit point of the sequence $(\mathbb{Q}_N^{\mu_N})_{N\geq2}$, referred to as $\mathbb{Q}_\infty$, is concentrated on the set $D^0([0,T];\mathcal{M})$ of couple of measure processes that are, at any time, absolutely continuous with respect to the Lebesgue measure on Λ and $\mathbb{T}^{p-1}$ respectively. More precisely, for any $\mathbb{Q}_\infty$ in the closure of $(\mathbb{Q}_N^{\mu_N})_{N\geq2}$, we have*

$$\mathbb{Q}_\infty\left(D^0([0,T];\mathcal{M})\right) = 1,$$

where

$$D^0([0,T];\mathcal{M}) := \Big\{\left(\pi^{\mathrm{field}}(t),\pi^{\mathrm{road}}(t)\right)_{t\in[0,T]} \in D([0,T];\mathcal{M}) \text{ such that,}$$

$$\text{for any } t\in[0,T], \left(\pi^{\mathrm{field}}(t),\pi^{\mathrm{road}}(t)\right) \text{ is absolutely}$$

$$\text{continuous with respect to the Lebesgue measure on}$$

$$(\Lambda\times\mathbb{T}^{p-1}), \text{ with density } (v(t,\cdot),u(t,\cdot))\in[0,1]^\Lambda\times[0,1]^{\mathbb{T}^{p-1}}\Big\}.$$

Indeed, the simple exclusion rule provides the upper bounds

$$|\langle \pi^{\text{field}}(t), G \rangle| \leq \|G\|_{L^1(\Lambda)} \qquad \text{and} \qquad |\langle \pi^{\text{road}}(t), H \rangle| \leq \|H\|_{L^1(\mathbb{T}^{p-1})}, \qquad (4.1)$$

for any $t \in [0, T]$, $G \in \mathcal{C}(\overline{\Lambda})$ and $H \in \mathcal{C}(\mathbb{T}^{p-1})$. Now, denoting λ the Lebesgue measure on Λ, let A be a Borel set such that $\lambda(A) = 0$. From a Corollary of Lusin's Theorem, see [48, subsection 2.24], there is a sequence of functions $(g_n)_{n \in \mathbb{N}}$ such that, for any $n \in \mathbb{N}$, $g_n \in \mathcal{C}_c^0(\Lambda)$, $|g_n| \leq 1$ and $\lim_{n+\to\infty} g_n(x) = \mathbf{1}_A(x)$ a.e. in Λ. We write

$$\pi^{\text{field}}(t)(A) = \langle \pi^{\text{field}}(t), \mathbf{1}_A \rangle = \langle \pi^{\text{field}}(t), \mathbf{1}_A - g_n \rangle_+ \langle \pi^{\text{field}}(t), g_n \rangle$$
$$\leq \langle \pi^{\text{field}}(t), \mathbf{1}_A - g_n \rangle_+ \int_\Lambda g_n(x) d\lambda(x),$$

thanks to (4.1). By the dominated convergence theorem, as $n \to +\infty$, the first term in the right hand side tends to zero, while the second term tends to $\lambda(A)$, so that $\pi^{\text{field}}(t)(A) \leq \lambda(A)$. Finally, since $\lambda(A) = 0$, we have $\pi^{\text{field}}(t)(A) = 0$ and $\pi^{\text{field}}(t)$ is thus absolutely continuous with respect to the Lebesgue measure. The same applies for $\pi^{\text{road}}(t)$.

4.2 The limit points of $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$ load paths whose densities satisfy (W1)

The next step consists in showing that the limit trajectories own the regularity claimed in (W1). Let us define the set $\mathscr{S}_{\text{W1}}$ by

$$\mathscr{S}_{\text{W1}} := \left\{ \left(\pi^{\text{field}}(t), \pi^{\text{road}}(t) \right)_{t \in [0,T]} = \left(v(t, \hat{x}) d\hat{x}, \ u(t, x) dx \right)_{t \in [0,T]} \ \middle| \ (v, u) \text{ satisfies condition (W1)} \right\}.$$

Then we have the following proposition.

Proposition 4.2 (Identification of the limit sets) *Let $\mathbb{Q}_\infty$ be a limit point of the sequence $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$. Then,*

$$\mathbb{Q}_\infty(\mathscr{S}_{\text{W1}}) = 1. \qquad (4.2)$$

The function $u(t, \cdot)$ being in $L^2(\mathbb{T}^{p-1})$ directly follows from the simple exclusion rule which forces u to be positive and bounded by 1. On the other hand, $v(t, \cdot)$ being in $\mathcal{H}^1(\Lambda)$ comes from the Riesz representation theorem combined with the following energy estimate.

Lemma 4.3 (Energy estimate) *Given $v \in L^2(0, T; L^2(\Lambda))$ and $1 \leq q \leq p$, consider the (potentially infinite) quantity*

$$E_q(v) := \sup_{G \in \mathcal{C}_c^{0,2}([0,T] \times \overline{\Lambda})} \left\{ \int_0^T \left\langle v(s), \partial_{e_q} G(s) \right\rangle_\Lambda ds - \frac{1}{2} \int_0^T \|G(s)\|_{L^2(\Lambda)}^2 ds \right\}. \qquad (4.3)$$

Then, for any $q \in [\![1; p]\!]$,

$$\mathbb{Q}_\infty \left(\left(\pi^{\text{field}}(t), \pi^{\text{road}}(t) \right)_{t \in [0,T]} = \left(v(t, \hat{x}) d\hat{x}, \ u(t, x) dx \right)_{t \in [0,T]} \ \middle| \ E_q(v) < +\infty \right) = 1.$$

A proof of Lemma 4.3 is given in Appendix A.3.

4.3 Replacement lemmas

Before obtaining the limit equations satisfied by the densities loaded by the limit points of $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$, we need to state two Replacement lemmas to correctly ensure the convergence of the boundary terms of the martingale $\mathscr{M}_N$ towards those of the weak formulation in (W2) — see subsection 4.4 for details. The essence of these lemmas lies in comparing the occupancy status of η at the boundary sites to the average number of particles in their immediate vicinity. To state such a result, we must define these "substitute objects". For fixed $\hat{\imath} \in \Gamma_N$ and $\varepsilon > 0$, let

$$\Lambda_{\hat{\imath}}^{\varepsilon N} := \left\{ \hat{\kappa} \in \Lambda_N : |\hat{\imath} - \hat{\kappa}| \leq \varepsilon N \right\} = \Lambda_N \cap \left\{ \hat{\imath} + [-\varepsilon N, \varepsilon N]^p \right\}, \tag{4.4}$$

and define for $\eta \in S_N^{\text{field}}$ the average number of particle of η inside the box $\Lambda_{\hat{\imath}}^{\varepsilon N}$, that is

$$\eta^{\varepsilon N}(\hat{\imath}) := c_{N,\varepsilon} \sum_{\hat{\kappa} \in \Lambda_{\hat{\imath}}^{\varepsilon N}} \eta(\hat{\kappa}), \tag{4.5}$$

where $c_{N,\varepsilon}$ is the number of sites inside $\Lambda_{\hat{\imath}}^{\varepsilon N}$, namely,

$$c_{N,\varepsilon} := \frac{1}{|\Lambda_{\hat{\imath}}^{\varepsilon N}|} = \left[\left(2\lfloor \varepsilon N \rfloor + 1 \right)^{p-1} \times \left(\lfloor \varepsilon N \rfloor + 1 \right) \right]^{-1}. \tag{4.6}$$

Let us note that this applies specifically to boundary points $\hat{\imath} \in \Gamma_N$, where the size of $\Lambda_{\hat{\imath}}^{\varepsilon N}$ is correctly captured by (4.6). The condition $\hat{\imath} = (i,j)$ with $j \in \{1, N-1\}$ is assumed here because we apply replacement lemmas exclusively at the boundary, making the averaging process relevant only for these points. For interior points, the formula would differ, as the structure of the neighborhood would change.

The upper Replacement lemma is the following.

Lemma 4.4 (Replacement at the upper boundary) *Let $(\mu_N)_{N \geq 2}$ be a sequence of initial measures on S_N. For any test function $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$, we have*

$$\limsup_{\varepsilon \to 0^+} \limsup_{N \to \infty} \mathbb{E}_N^{\mu_N} \left[\left| \int_0^T \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Gamma_N^{\text{up}}} G(s, \tfrac{\hat{\imath}}{N}) \left[\eta_s^{\varepsilon N}(\hat{\imath}) - \eta_s(\hat{\imath}) \right] ds \right| \right] = 0. \tag{4.7}$$

Similarly, the lower Replacement lemma is the following.

Lemma 4.5 (Replacement at the lower boundary) *Let $(\mu_N)_{N \geq 2}$ be a sequence of initial measures on S_N. For any test function $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$, we have*

$$\limsup_{\varepsilon \to 0^+} \limsup_{N \to \infty} \mathbb{E}_N^{\mu_N} \left[\left| \int_0^T \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Gamma_N^{\text{low}}} G(s, \tfrac{\hat{\imath}}{N}) \left[\eta_s^{\varepsilon N}(\hat{\imath}) - \eta_s(\hat{\imath}) \right] ds \right| \right] = 0. \tag{4.8}$$

The proof of the Lemma 4.4 is given in Appendix A.2, while that of Lemma 4.5 is omitted since it follows essentially identical arguments.
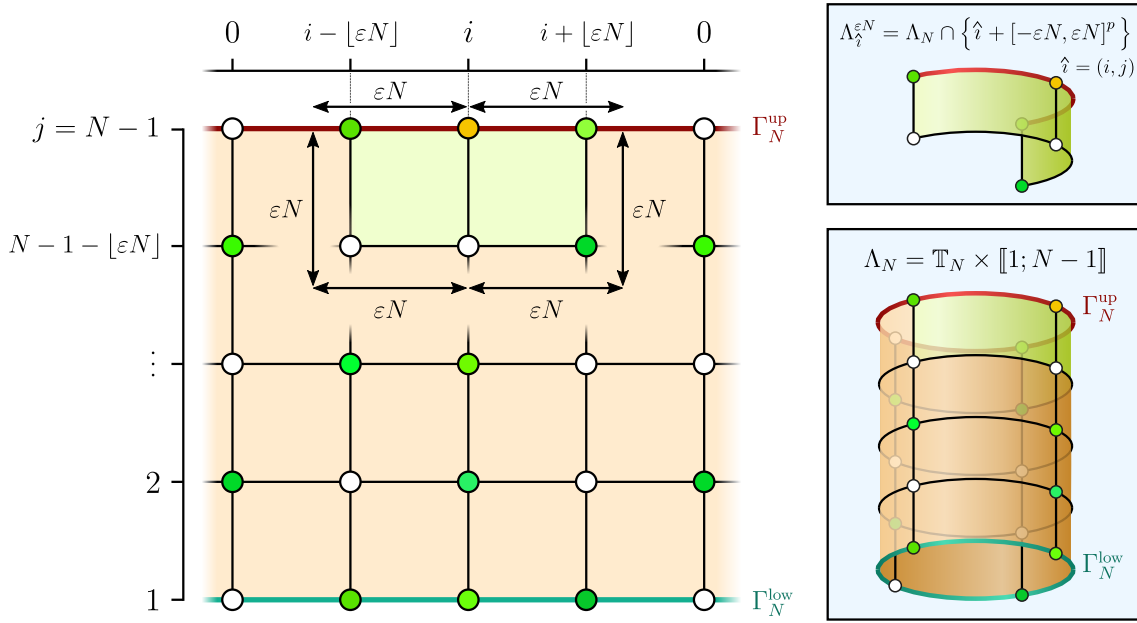
Figure II — The upper Replacement lemma. *For $\hat{\imath} = (i, j) \in \Gamma_N^{\mathrm{up}}$, we define $\Lambda_{\hat{\imath}}^{\varepsilon N}$ as the intersection between Λ_N and the p-dimensional box of range $2\varepsilon N$ centered on $\hat{\imath}$. Notice that for fixed $\varepsilon > 0$, the size of the box remains constant at the macroscopic scale. Lemma 4.4 establishes that, as $N \to \infty$ followed by $\varepsilon \to 0^+$, the value of $\eta(\hat{\imath})$ can be replaced by $\eta^{\varepsilon N}(\hat{\imath})$ which is the mean value of η inside $\Lambda_{\hat{\imath}}^{\varepsilon N}$.*

4.4 The limit points of $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$ load paths whose densities satisfy (W2)

We claim now that all the limit trajectories are supported on the set $\mathscr{S}_{\mathrm{W2}}$ of measures with density (v, u) that satisfy the weak formulation associated with (1.2)-(1.3), namely

$$\mathscr{S}_{\mathrm{W2}} := \left\{ \left(\pi^{\mathrm{field}}(t), \pi^{\mathrm{road}}(t) \right)_{t \in [0,T]} = \left(v(t, \hat{x}) d\hat{x}, \ u(t, x) dx \right)_{t \in [0,T]} \ \middle| \ (v, u) \text{ satisfies condition (W2)} \right\}.$$

Proposition 4.6 (Identification of the limit equations) *Let $\mathbb{Q}_\infty$ be a limit point of the sequence $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$. Then,*

$$\mathbb{Q}_\infty(\mathscr{S}_{\mathrm{W2}}) = 1. \tag{4.9}$$

To establish Proposition 4.6, we need to demonstrate that each component of the martingale $\mathscr{M}_N = \mathscr{M}_{N,G}^{\mathrm{field}} + \mathscr{M}_{N,H}^{\mathrm{road}}$, as detailed in (3.5)-(3.6), converges towards its respective counterpart in the weak formulation (2.7)-(2.8), and then that the martingale itself vanishes as $N \to \infty$. This requires the two Replacement lemmas (Lemma 4.4 and Lemma 4.5), which allow the substitution of terms not expressed through the empirical measure π_N in the martingale $\mathscr{M}_N$, and the identification of the limit sets (Proposition 4.2), that is essential to define the trace of the function $v(t, \cdot)$ at the lower and upper boundaries.

Proof of Proposition 4.6. Let $G \in \mathcal{C}^{1,2}([0, T] \times \overline{\Lambda})$ and $H \in \mathcal{C}^{1,2}([0, T] \times \mathbb{T}^{p-1})$. For practical reasons, let us define here the functional $\mathscr{W}_{u,v,(G,H)}(T) := \mathscr{W}_{u,v,G}^{\mathrm{field}}(T) + \mathscr{W}_{u,v,H}^{\mathrm{road}}(T)$

associated with the weak formulation (2.7)-(2.8) in (W2):

$$\mathscr{W}_{u,v,G}^{\text{field}}(T) := \langle v(T), G(T)\rangle_\Lambda - \langle v_0, G(0)\rangle_\Lambda - \int_0^T \langle v(s), \partial_s G(s)\rangle_\Lambda \, ds - \int_0^T \langle v(s), d\Delta G(s)\rangle_\Lambda \, ds$$

$$+ \int_0^T \langle v|_{y=1}(s), d\partial_y G|_{y=1}(s)\rangle_{\mathbb{T}^{p-1}} \, ds - \int_0^T \langle v|_{y=0}(s), d\partial_y G|_{y=0}(s)\rangle_{\mathbb{T}^{p-1}} \, ds \qquad (4.10)$$

$$- \int_0^T \alpha \langle u(s) - v|_{y=0}(s), G|_{y=0}(s)\rangle_{\mathbb{T}^{p-1}} \, ds,$$

and

$$\mathscr{W}_{u,v,H}^{\text{road}}(T) := \langle u(T), H(T)\rangle_{\mathbb{T}^{p-1}} - \langle u_0, H(0)\rangle_{\mathbb{T}^{p-1}} - \int_0^T \langle u(s), \partial_s H(s)\rangle_{\mathbb{T}^{p-1}} \, ds$$

$$\qquad (4.11)$$

$$- \int_0^T \langle u(s), D\Delta_x H(s)\rangle_{\mathbb{T}^{p-1}} \, ds - \int_0^T \alpha \langle v|_{y=0}(s) - u(s), H(s)\rangle_{\mathbb{T}^{p-1}} \, ds.$$

Given $\mathbb{Q}_\infty$ in the closure of the sequence $(\mathbb{Q}_N^{\mu_N})_{N \geq 2}$, the statement of Proposition 4.6 can then be reformulated as

$$\mathbb{Q}_\infty\left(\left(\pi^{\text{field}}(t), \pi^{\text{road}}(t)\right)_{t \in [0,T]} \,\bigg|\, \pi^{\text{field}}(t) = v(t, \hat{x})d\hat{x}, \ \pi^{\text{road}}(t) = u(t, x)dx \ \text{ and}\right.$$

$$\left. \forall (G, H) \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda}) \times \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^{p-1}), \mathscr{W}_{u,v,G}^{\text{field}}(T) = \mathscr{W}_{u,v,H}^{\text{road}}(T) = 0 \right) = 1.$$

To prove this, it is sufficient to establish that, for any $\delta > 0$, any $(G, H) \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda}) \times \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^{p-1})$,

$$\mathbb{Q}_\infty\left(\left|\mathscr{W}_{u,v,(G,H)}(T)\right| > \delta \right) = 0. \qquad (4.12)$$

Indeed, assume (4.12) holds. By separability, there is a sequence $(G_n, H_n)_{n \in \mathbb{N}}$ which is dense in $\mathcal{C}^{1,2}([0,T] \times \overline{\Lambda}) \times \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^{p-1})$. Then, for any $K \in \mathbb{N}$, we have

$$\mathbb{Q}_\infty\left(\sup_{0 \leq n \leq K} \left|\mathscr{W}_{u,v,(G_n,H_n)}(T)\right| > \delta \right) \leq \sum_{n=0}^K \mathbb{Q}_\infty\left(\left|\mathscr{W}_{u,v,(G_n,H_n)}(T)\right| > \delta \right) = 0,$$

and conclude by monotone convergence Theorem.

We now turn to the proof of (4.12). At this point, to work with the probability measures $\mathbb{Q}_N^{\mu_N}$ (that do not load paths with densities) instead of $\mathbb{Q}_\infty$, we need to substitute the quantity $\mathscr{W}_{u,v,(G,H)}(T)$ with another one which only depends on measures π^{field} and π^{road} instead of v and u. This substitution cannot be made directly because of the boundary terms in (4.10) and (4.11). To overcome this issue, let us introduce the two families of unit approximations indexed by $\varepsilon > 0$,

$$\mathbb{U}_\varepsilon^{\text{up}} := \frac{1}{\varepsilon(2\varepsilon)^{p-1}} \mathbb{1}_{[-\varepsilon,\varepsilon]^{p-1} \times [0,\varepsilon]} \quad \text{and} \quad \mathbb{U}_\varepsilon^{\text{low}} := \frac{1}{\varepsilon(2\varepsilon)^{p-1}} \mathbb{1}_{[-\varepsilon,\varepsilon]^{p-1} \times [-\varepsilon,0]},$$

defined for all $x \in \mathbb{T}^{p-1}$ and $y \in [0,1]$. Observe that $\eta^{\varepsilon N}(\hat{\imath})$ defined in (4.5) can then be rewritten for any $\hat{\imath} \in \Gamma_N = \Gamma_N^{\text{up}} \cup \Gamma_N^{\text{low}}$, $\varepsilon > 0$ and $N \geq 2$ as

$$\eta^{\varepsilon N}(\hat{\imath}) = \tilde{c}_{N,\varepsilon} \left[\pi_N^{\text{field}} * \mathbb{U}_\varepsilon^{\text{up}}\right](\hat{\imath}/N), \qquad \text{if } \hat{\imath} \in \Gamma_N^{\text{up}}, \qquad (4.13)$$

and

$$\eta^{\varepsilon N}(\hat{\imath}) = \tilde{c}_{N,\varepsilon} \, [\pi_N^{\text{field}} * \mathbb{U}_\varepsilon^{\text{low}}](\hat{\imath}/N), \qquad \text{if } \hat{\imath} \in \Gamma_N^{\text{low}}, \tag{4.14}$$

where

$$\tilde{c}_{N,\varepsilon} = \frac{(2\varepsilon N)^{p-1}(\varepsilon N)}{(2\lfloor \varepsilon N \rfloor + 1)^{p-1}(\lfloor \varepsilon N \rfloor + 1)} \quad \text{for any } \varepsilon > 0 \text{ and any } N \geq 2,$$

and the convolutions of the unit approximations $\mathbb{U}_\varepsilon^{\text{up}}$ and $\mathbb{U}_\varepsilon^{\text{low}}$ with the Radon measure π_N^{field} are well-defined functions on Λ given, for any $\hat{\hat{x}} \in \Lambda$, by

$$[\pi_N^{\text{field}} * \mathbb{U}_\varepsilon^{\text{up}}](\hat{\hat{x}}) = \int_\Lambda \mathbb{U}_\varepsilon^{\text{up}}(\hat{\hat{x}} - \hat{\hat{z}}) \, d\pi_N^{\text{field}}(\hat{\hat{z}}) \quad \text{and} \quad [\pi_N^{\text{field}} * \mathbb{U}_\varepsilon^{\text{low}}](\hat{\hat{x}}) = \int_\Lambda \mathbb{U}_\varepsilon^{\text{low}}(\hat{\hat{x}} - \hat{\hat{z}}) \, d\pi_N^{\text{field}}(\hat{\hat{z}})$$

respectively. Now for $\varepsilon > 0$, $\pi = (\pi^{\text{field}}, \pi^{\text{road}}) \in \mathcal{M}$, and $t \in [0, T]$, we let $\mathscr{W}_{\pi,(G,H)}^\varepsilon(T) := \mathscr{W}_{\pi,G,\varepsilon}^{\text{field}}(T) + \mathscr{W}_{\pi,H,\varepsilon}^{\text{road}}(T)$ with

$$\mathscr{W}_{\pi,G,\varepsilon}^{\text{field}}(T) := \langle \pi^{\text{field}}(T), G(T) \rangle - \langle \pi^{\text{field}}(0), G(0) \rangle - \int_0^T \langle \pi^{\text{field}}(s), \partial_s G(s) \rangle ds - \int_0^T \langle \pi^{\text{field}}(s), d\Delta G(s) \rangle ds$$

$$+ \int_0^T \left\langle [\pi^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{up}}]|_{y=1}, d\partial_y G|_{y=1}(s) \right\rangle_{\mathbb{T}^{p-1}} ds - \int_0^T \left\langle [\pi^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{y=0}, d\partial_y G|_{y=0}(s) \right\rangle_{\mathbb{T}^{p-1}} ds$$

$$- \int_0^T \alpha \langle \pi^{\text{road}}(s), G|_{y=0}(s) \rangle + \int_0^T \alpha \left\langle [\pi^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{y=0}, G|_{y=0}(s) \right\rangle_{\mathbb{T}^{p-1}} ds, \tag{4.15}$$

and

$$\mathscr{W}_{\pi,H,\varepsilon}^{\text{road}}(T) := \langle \pi^{\text{road}}(T), H(T) \rangle - \langle \pi^{\text{road}}(0), H(0) \rangle - \int_0^T \langle \pi^{\text{road}}(s), \partial_s H(s) \rangle ds$$

$$- \int_0^T \langle \pi^{\text{road}}(s), D\Delta_x H(s) \rangle ds - \int_0^T \alpha \left\langle [\pi^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{y=0}, H(s) \right\rangle_{\mathbb{T}^{p-1}} + \int_0^T \alpha \langle \pi^{\text{road}}(s), H(s) \rangle ds. \tag{4.16}$$

Thanks to Proposition 4.2, we know that $\mathbb{Q}_\infty$ loads paths with densities $(v(t,\cdot))_{t \in [0,T]}$ in $L^2(0, T; \mathcal{H}^1(\Lambda))$. As a consequence, the trace of $v(t,\cdot)$ at the boundaries Γ^{up} and Γ^{low} is well-defined and we have (see [25, Section 5.3])

$$\lim_{\varepsilon \to 0^+} [\pi^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{up}}]|_{\hat{\hat{x}} \in \Gamma^{\text{up}}} = v|_{y=1}(s), \qquad \mathbb{Q}_\infty\text{-almost-surely in } \Gamma^{\text{up}},$$

and

$$\lim_{\varepsilon \to 0^+} [\pi^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{\hat{\hat{x}} \in \Gamma^{\text{low}}} = v|_{y=0}(s), \qquad \mathbb{Q}_\infty\text{-almost-surely in } \Gamma^{\text{low}}.$$

As a result of this

$$\lim_{\varepsilon \to 0^+} \left| \mathscr{W}_{u,v,(G,H)}(T) - \mathscr{W}_{\pi,(G,H)}^\varepsilon(T) \right| = 0, \qquad \mathbb{Q}_\infty\text{-almost-surely.} \tag{4.17}$$

We can now bound the probability in (4.12) as follows

$$\mathbb{Q}_\infty \left(\left| \mathscr{W}_{u,v,(G,H)}(T) \right| > \delta \right) = \mathbb{Q}_\infty \left(\left| \mathscr{W}_{u,v,(G,H)}(T) - \mathscr{W}_{\pi,(G,H)}^\varepsilon(T) + \mathscr{W}_{\pi,(G,H)}^\varepsilon(T) \right| > \delta \right)$$

$$\leq \mathbb{Q}_\infty \left(\left| \mathscr{W}_{u,v,(G,H)}(T) - \mathscr{W}_{\pi,(G,H)}^\varepsilon(T) \right| > \delta/2 \right) + \mathbb{Q}_\infty \left(\left| \mathscr{W}_{\pi,(G,H)}^\varepsilon(T) \right| > \delta/2 \right)$$

$$\leq \mathbb{Q}_\infty \left(\left| \mathscr{W}_{u,v,(G,H)}(T) - \mathscr{W}_{\pi,(G,H)}^\varepsilon(T) \right| > \delta/2 \right) + \liminf_{N \to \infty} \mathbb{Q}_N^{\mu_N} \left(\left| \mathscr{W}_{\pi_N,(G,H)}^\varepsilon(T) \right| > \delta/2 \right), \tag{4.18}$$

where we used the Portmanteau Theorem to write the last inequality. While the vanishing of the first term in (4.18) as $\varepsilon \to 0^+$ is a straight consequence of (4.17), the second one requires further attention. By considering the probability under the limit in this second term, we control it by this way:

$$\mathbb{Q}_N^{\mu_N}\left(\left|\mathscr{W}_{\pi_N,(G,H)}^\varepsilon(T)\right| > \delta/2\right) = \mathbb{Q}_N^{\mu_N}\left(\left|\mathscr{W}_{\pi_N,(G,H)}^\varepsilon(T) - \mathscr{M}_N(T) + \mathscr{M}_N(T)\right| > \delta/2\right)$$

$$\leq \mathbb{Q}_N^{\mu_N}\left(\left|\mathscr{W}_{\pi_N,(G,H)}^\varepsilon(T) - \mathscr{M}_N(T)\right| > \delta/4\right) + \mathbb{Q}_N^{\mu_N}\left(\left|\mathscr{M}_N(T)\right| > \delta/4\right),$$

$$\leq \frac{4}{\delta}\mathbb{E}_N^{\mu_N}\left(\left|\mathscr{W}_{\pi_N,(G,H)}^\varepsilon(T) - \mathscr{M}_N(T)\right|\right) + \mathbb{Q}_N^{\mu_N}\left(\left|\mathscr{M}_N(T)\right| > \delta/4\right), \tag{4.19}$$

where we recall that the martingale $\mathscr{M}_N(T)$ is defined in (3.1). In (4.19), the vanishing of $\mathbb{Q}_N^{\mu_N}(|\mathscr{M}_N(T)| > \delta/4)$ can be shown by using the Doob's inequality:

$$\mathbb{Q}_N^{\mu_N}\left(\left|\mathscr{M}_N(T)\right| > \delta/4\right) \leq \mathbb{Q}_N^{\mu_N}\left(\sup_{0 \leq t \leq T}\left|\mathscr{M}_N(t)\right| > \delta/4\right) \leq \frac{16}{\delta^2}\mathbb{E}_N^{\mu_N}\left(\left[\mathscr{M}_N(T)\right]^2\right)$$

$$\stackrel{(3.3)}{=} \frac{16}{\delta^2}\mathbb{E}_N^{\mu_N}\left(\mathscr{N}_N(T) + \int_0^T \mathscr{B}_N(s)ds\right)$$

$$= \frac{16}{\delta^2}\mathbb{E}_N^{\mu_N}\left(\int_0^T \mathscr{B}_N(s)ds\right),$$

that goes to zero as $N \to \infty$ — see the control of $\mathscr{B}_N(s)$ below (3.16). Now we focus on the remaining expectation in (4.19), and note that the quantity $|\mathscr{W}_{\pi_N,(G,H)}^\varepsilon(T) - \mathscr{M}_N(T)|$ can be bounded by the sum of the following terms

$$\left|\int_0^T \langle \pi_N^{\text{field}}(s), d(\Delta - \Delta_x^N - \partial_{yy}^N)G(s)\rangle + O(1/N)\, ds\right|, \tag{4.20}$$

$$\left|\int_0^T \left\langle [\pi_N^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{up}}]|_{y=1}, d\partial_y G|_{y=1}(s)\right\rangle_{\mathbb{T}^{p-1}} - \frac{1}{N^{p-1}}\sum_{i \in \Gamma_N^{\text{up}}} d\partial_y^N G(s, \tfrac{i}{N}) \times \eta_s(i)\, ds\right|, \tag{4.21}$$

$$\left|\int_0^T \left\langle [\pi_N^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{y=0}, d\partial_y G|_{y=0}(s)\right\rangle_{\mathbb{T}^{p-1}} - \frac{1}{N^{p-1}}\sum_{i \in \Gamma_N^{\text{low}}} d\partial_y^N G(s, \tfrac{i}{N}) \times \eta_s(i)\, ds\right|, \tag{4.22}$$

$$\left|\int_0^T \alpha \left\langle [\pi_N^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{y=0}, G|_{y=0}(s)\right\rangle_{\mathbb{T}^{p-1}} - \frac{\alpha}{N^{p-1}}\sum_{i \in \Gamma_N^{\text{low}}} G(s, \tfrac{i}{N}) \times \eta_s(i)\, ds\right|, \tag{4.23}$$

$$\left|\int_0^T \alpha \langle \pi_N^{\text{road}}(s), G|_{y=0}(s) - G|_{y=\frac{1}{N}}(s)\rangle\, ds\right|, \tag{4.24}$$

$$\left|\int_0^T \frac{1}{N^p}\sum_{i \in \Gamma_N^{\text{up}}} G(s, \tfrac{i}{N}) \times (b - \eta_s(i))ds\right|, \tag{4.25}$$

for the "field parts", and

$$\left| \int_0^T \langle \pi_N^{\text{road}}(s), D(\Delta_x - \Delta_x^N)H(s) \rangle \, ds \right|, \tag{4.26}$$

$$\left| \int_0^T \alpha \left\langle [\pi_N^{\text{field}}(s) * \mathbb{U}_\varepsilon^{\text{low}}]|_{y=0}, H(s) \right\rangle_{\mathbb{T}^{p-1}} - \frac{\alpha}{N^{p-1}} \sum_{i \in \Gamma_N^{\text{low}}} H(s, \tfrac{i}{N}) \times \eta_s(i) \, ds \right|, \tag{4.27}$$

for the "road parts". To conclude this proof, it remains to argue that all these terms (4.20)-(4.27) vanish in the limit $N \to \infty$ then $\varepsilon \to 0^+$. First of all, due to the regularity of the test functions G and H, it is clear that (4.20), (4.24) and (4.26) go to zero as $N \to \infty$. Notice then that (4.25) associated to the upper spawn/kill dynamics is a $O(1/N)$. Lastly, the vanishing of (4.21)-(4.22)-(4.23)-(4.27) arises from the Replacement lemmas (Lemma 4.4 and Lemma 4.5), the approximation of $\eta_s^{\varepsilon N}(i)$ with the convolution products (4.13) and (4.14), and the regularity of the test functions G and H. $\qquad\square$

5 Uniqueness of the solution

In this last section, we establish the uniqueness of the weak solutions to the Cauchy problem (1.2)-(1.3) in the sense (W1)-(W2). Our proof relies on testing the weak formulation (W2) against the solutions (G, H) to a "dual problem" related to (1.2).

Proposition 5.1 (Uniqueness) *There exists at most one solution (v, u) to the Cauchy problem* (1.2)-(1.3) *in the sense* (W1)-(W2).

Proof of Proposition 5.1 (Uniqueness). By linearity it is enough to consider the case $(v_0, u_0) \equiv (0, 0)$. Given any $\varphi \in \mathcal{C}_c^\infty([0, T] \times \Lambda)$ and any $\psi \in \mathcal{C}_c^\infty([0, T] \times \mathbb{T}^{p-1})$, we consider the problem

$$\begin{cases} -\partial_t G - d\Delta G = \varphi, & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \quad y \in (0, 1), \\ -d\partial_y G|_{y=0} = \alpha(H - G|_{y=0}), & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \quad y = 0, \\ -\partial_t H - D\Delta_x H - \alpha(G|_{y=0} - H) = \psi, & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \\ \partial_y G|_{y=1} = 0, & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \quad y = 1, \end{cases} \tag{5.1}$$

supplemented with final condition

$$\begin{cases} G|_{t=T} \equiv 0, & x \in \mathbb{T}^{p-1}, \quad y \in (0, 1), \\ H|_{t=T} \equiv 0, & x \in \mathbb{T}^{p-1}. \end{cases} \tag{5.2}$$

By letting $(\tilde{G}(t), \tilde{H}(t)) = (G(T - t), H(T - t))$, we can notice that the problem (5.1)-(5.2) is actually a reversed-time field-road system with sources, namely

$$\begin{cases} \partial_t \tilde{G} = d\Delta \tilde{G} + \varphi, & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \quad y \in (0, 1), \\ -d\partial_y \tilde{G}|_{y=0} = \alpha(\tilde{H} - \tilde{G}|_{y=0}), & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \quad y = 0, \\ \partial_t \tilde{H} = D\Delta_x \tilde{H} + \alpha(\tilde{G}|_{y=0} - \tilde{H}) + \psi, & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \\ \partial_y \tilde{G}|_{y=1} = 0, & t \in (0, T), \quad x \in \mathbb{T}^{p-1}, \quad y = 1, \end{cases} \tag{5.3}$$

provided with trivial initial data. In absence of sources, the solution is known through the "field-road heat kernel", that could be computed thanks to an adaptation of the technics in [3], see Remark 5.2. By combining this with the Duhamel principle, see [32, Chapter 4, Section 3] for instance, we obtain the classical solution to the above problem with sources. Note also that so-called "reaction-diffusion arguments" may also be applied, see Remark 5.2 again. As a result, we own $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$ and $H \in \mathcal{C}^{1,2}([0,T] \times \mathbb{T}^{p-1})$ that satisfy (5.1)-(5.2), and that are sufficiently smooth to be tested in the weak formulation (W2). By plugging (G, H) into (2.7) and (2.8), and then summing the two obtained results, we are left with

$$\int_0^T \langle v(s), \varphi(s) \rangle_\Lambda \, ds + \int_0^T \langle u(s), \psi(s) \rangle_{\mathbb{T}^{p-1}} \, ds = 0,$$

that holds for any $\varphi \in \mathcal{C}_c^\infty([0,T] \times \Lambda)$ and any $\psi \in \mathcal{C}_c^\infty([0,T] \times \mathbb{T}^{p-1})$. In particular we have

$$\begin{cases} \forall \varphi \in \mathcal{C}_c^\infty([0,T] \times \Lambda), & \int_0^T \langle v(s), \varphi(s) \rangle_\Lambda \, ds = 0, \\ \forall \psi \in \mathcal{C}_c^\infty([0,T] \times \mathbb{T}^{p-1}), & \int_0^T \langle u(s), \psi(s) \rangle_{\mathbb{T}^{p-1}} \, ds = 0. \end{cases} \tag{5.4}$$

From (5.4), we can deduce that v and u are both identically zero, see [17, Lemma IV.2], and the proof is therefore completed. $\qquad\square$

Remark 5.2 *Let us recall that, originally, the purely diffusive field-road model, obtained by letting $f \equiv 0$ in (1.1), was posed in the half-space $\mathbb{R}^{p-1} \times (0, +\infty)$. Very recently, an explicit expression for its fundamental solution, and for the solution to the associated Cauchy problem, was obtained in [3]. The main ingredient is to apply a "Fourier on the road variable $x \in \mathbb{R}^{p-1}$/Laplace on the time variable $t \in (0, +\infty)$ transform" to reach a linear second order ODE for the Fourier/Laplace transform of the solution which is function of $y \in (0, +\infty)$ only (the Fourier variables $\xi \in \mathbb{R}^{p-1}$ and the Laplace variable $s \in (0, +\infty)$ serving as parameters), see equation (4.3) in [3]. This ODE is supplemented by the Fourier/Laplace transform of the Exchange condition at $y = 0$ and by the vanishing condition as $y \to +\infty$. After that, it "remains" to solve the ODE and to semi-implicitly invert the Fourier/Laplace transform to obtain the solution.*

In the present work, the purey diffusive field-road model is posed on $\mathbb{T}^{p-1} \times (0, 1)$. Because of that, one should use the Fourier series to deal with the variable $x \in \mathbb{T}^{p-1}$ (in place of the Fourier transform) and, still, the Laplace transform to deal with the variable $t \in (0, +\infty)$. After that, one would reach a linear second order ODE for a function of $y \in (0, 1)$ only (the Fourier indexes $n \in \mathbb{Z}^{p-1}$ and the Laplace variable $s \in (0, +\infty)$ serving as parameters). This ODE would be supplemented by the adequate transform of the Exchange condition at $y = 0$, as in [3], and a Neumann boundary condition at $y = 1$, in contrast with [3]. Such computations would deserve a lengthy analysis, are out the scope of the present work, but the path is well mapped and we are very confident on the outcome.

Note also that another possibility would be to adapt the so-called "reaction-diffusion arguments" (parabolic regularity, comparison principle, etc.) used in [10, Proposition 3.1] for the field-road model posed in the half-space.

Appendix

A.1 Some tools and basic estimates

To prove the upper Replacement lemma (Lemma 4.4) and the energy estimate (Lemma 4.3), we first need to introduce the relative entropy $\mathcal{H}$ and the Dirichlet form $\mathcal{D}_N$. These notions are rather classical and can be found in [34, Appendix 1, Sections 7-8-9-10] for instance.

For $\gamma \in (0,1)$, we denote by $\nu_N = \nu_{N,\gamma}^{\text{field}} \otimes \nu_{N,\gamma}^{\text{road}}$ the Bernoulli product measure on S_N whose marginals are given by

$$\nu_{N,\gamma}^{\text{field}}\Big[\eta(\hat{\imath}) = \bullet\Big] = \begin{cases} \gamma & \text{if } \bullet = 1, \\ 1 - \gamma & \text{if } \bullet = 0, \end{cases} \qquad \text{for any } \hat{\imath} \in \Lambda_N, \text{ and}$$

$$\nu_{N,\gamma}^{\text{road}}\Big[\xi(i) = \bullet\Big] = \begin{cases} \gamma & \text{if } \bullet = 1, \\ 1 - \gamma & \text{if } \bullet = 0, \end{cases} \qquad \text{for any } i \in \Gamma_N^{\text{low}}.$$

The probability measure ν_N on S_N offers interesting properties to work with the relative entropy and the Dirichlet form. Indeed, since ν_N gives a positive probability to each configuration $(\eta, \xi) \in S_N$, any probability measure μ_N on S_N is absolutely continuous with respect to ν_N. Moreover, the changes of variable of type "flip" or "switch" as done below are very simple to express when integrating with respect to ν_N.

For a probability measure μ_N on S_N, the entropy of μ_N with respect to ν_N is defined as the positive value

$$\mathcal{H}\big(\mu_N | \nu_N\big) := \sup_{f \in \mathbb{R}^{S_N}} \left\{ \int_{S_N} f(\eta, \xi)\, d\mu_N(\eta, \xi) - \log\left[\int_{S_N} e^{f(\eta,\xi)}\, d\nu_N(\eta, \xi)\right] \right\}.$$

Since μ_N is absolutely continuous with respect to ν_N, the entropy can be explicitly written as [34, Appendix 1, Sections 8, Theorem 8.3]

$$\mathcal{H}\big(\mu_N | \nu_N\big) = \int_{S_N} \log\left[\frac{d\mu_N}{d\nu_N}(\eta, \xi)\right] d\mu_N(\eta, \xi) = \sum_{(\eta,\xi) \in S_N} \mu_N(\eta, \xi) \log\left[\frac{\mu_N(\eta, \xi)}{\nu_N(\eta, \xi)}\right],$$

where $d\mu_N / d\nu_N$ denotes the Radon-Nikodym derivative of μ_N with respect to ν_N and the last equality is a consequence of the finiteness of S_N. By decomposing μ_N as a convex combination of Dirac masses and using the convexity of the entropy, we can show that

$$\mathcal{H}\big(\mu_N | \nu_N\big) \leq C_0 N^p, \tag{A.1}$$

for $C_0 := -\log(\min(\gamma, 1 - \gamma)) > 0$.

Given a function $f : S_N \to \mathbb{R}$, the Dirichlet form associated to the dynamics can be defined as

$$\mathcal{D}_N(f, \nu_N) := -\langle \mathcal{L}_N f, f \rangle_{\nu_N} \tag{A.2}$$

and may be split following the different parts of $\mathcal{L}_N$ — see (2.1) — into

$$\mathcal{D}_N = N^2\, \mathbb{D}_N^{\text{field}} + N^2\, \mathbb{D}_N^{\text{road}} + N\, D_N^{\text{Rob}} + D_N^{\text{reac}} + D_N^{\text{up}}. \tag{A.3}$$

At some points in the proofs of upper Replacement lemma (Lemma 4.4) and the energy estimate (Lemma 4.3), it becomes essential to control each component of $\mathcal{D}_N$. These controls are encapsulated in Lemma A.1 below, and require to introduce the functional

$$\mathcal{I}_N = \mathcal{I}_N(f, \nu_N) := N^2\, \mathbb{I}_N^{\text{field}} + N^2\, \mathbb{I}_N^{\text{road}} + N\, I_N^{\text{Rob}} + I_N^{\text{reac}} + I_N^{\text{up}}, \tag{A.4}$$

where, for any $f : S_N \to \mathbb{R}$,

$$\mathbb{I}_N^{\text{field}}(f, \nu_N) := \frac{d}{2} \sum_{\substack{\hat{\imath}, \hat{k} \in \Lambda_N \\ |\hat{\imath} - \hat{k}| = 1}} \int_{S_N} \left[\sqrt{f(\eta^{\hat{\imath}, \hat{k}}, \xi)} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi),$$

$$\mathbb{I}_N^{\text{road}}(f, \nu_N) := \frac{D}{2} \sum_{\substack{i, k \in \Gamma_N^{\text{low}} \\ |i - k| = 1}} \int_{S_N} \left[\sqrt{f(\eta, \xi^{i,k})} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi),$$

$$I_N^{\text{Rob}}(f, \nu_N) := \alpha \sum_{i \in \Gamma_N^{\text{low}}} \int_{S_N} \left(\eta(i) - \xi(i) \right)^2 \left[\sqrt{f(\eta^i, \xi)} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi),$$

$$I_N^{\text{reac}}(f, \nu_N) := \alpha \sum_{i \in \Gamma_N^{\text{low}}} \int_{S_N} \left(\eta(i) - \xi(i) \right)^2 \left[\sqrt{f(\eta, \xi^i)} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi),$$

$$I_N^{\text{up}}(f, \nu_N) := \sum_{i \in \Gamma_N^{\text{up}}} \int_{S_N} \left(b(1 - \eta(i)) + (1 - b)\eta(i) \right) \left[\sqrt{f(\eta^i, \xi)} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi),$$

are all nonnegative.

Lemma A.1 (Control of the Dirichlet forms) *For any density function $f : S_N \to \mathbb{R}$ with respect to the Bernoulli product measure ν_N on S_N with parameter γ, we have*

(D1) *For any $\gamma \in (0, 1)$,*

$$\langle \mathbb{L}_N^{\text{field}} \sqrt{f}, \sqrt{f} \rangle_{\nu_N} = -\mathbb{D}_N^{\text{field}}(\sqrt{f}, \nu_N) = -\frac{1}{2}\mathbb{I}_N^{\text{field}}(f, \nu_N). \tag{A.5}$$

(D2) *For any $\gamma \in (0, 1)$,*

$$\langle \mathbb{L}_N^{\text{road}} \sqrt{f}, \sqrt{f} \rangle_{\nu_N} = -\mathbb{D}_N^{\text{road}}(\sqrt{f}, \nu_N) = -\frac{1}{2}\mathbb{I}_N^{\text{road}}(f, \nu_N). \tag{A.6}$$

(D3) *For any $\gamma \in (0, 1)$,*

$$\langle L_N^{\text{Rob}} \sqrt{f}, \sqrt{f} \rangle_{\nu_N} = -D_N^{\text{Rob}}(\sqrt{f}, \nu_N) = -\frac{1}{2}I_N^{\text{Rob}}(f, \nu_N) + \mathcal{E}_N(\alpha, \gamma, f). \tag{A.7}$$

(D4) *For any $\gamma \in (0,1)$,*

$$\langle L_N^{\mathrm{reac}} \sqrt{f}, \sqrt{f} \rangle_{\nu_N} = -D_N^{\mathrm{reac}}(\sqrt{f}, \nu_N) = -\frac{1}{2} I_N^{\mathrm{reac}}(f, \nu_N) + \mathcal{E}_N(\alpha, \gamma, f). \qquad (A.8)$$

(D5) *For $\gamma = b$,*

$$\langle L_N^{\mathrm{up}} \sqrt{f}, \sqrt{f} \rangle_{\nu_N} = -D_N^{\mathrm{up}}(\sqrt{f}, \nu_N) = -\frac{1}{2} I_N^{\mathrm{up}}(f, \nu_N). \qquad (A.9)$$

In (D3) and (D4), $\mathcal{E}_N(\alpha, \gamma, f)$ is twice the same quantity, and such that

$$|\mathcal{E}_N(\alpha, \gamma, f)| \leq c(\gamma) \alpha N^{p-1}. \qquad (A.10)$$

Proof of Lemma A.1, (D1). By writing $\mathbb{D}_N^{\mathrm{field}}$ from (A.2) and (2.2), we get

$$-\mathbb{D}_N^{\mathrm{field}}(\sqrt{f}, \nu_N) = \frac{d}{4} \sum_{\substack{\hat{\imath}, \hat{\kappa} \in \Lambda_N \\ |\hat{\imath} - \hat{\kappa}| = 1}} \int_{S_N} \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\kappa}}, \xi)} - \sqrt{f(\eta, \xi)} \right] \sqrt{f(\eta, \xi)} \, d\nu_N(\eta, \xi)$$

$$+ \frac{d}{4} \sum_{\substack{\hat{\imath}, \hat{\kappa} \in \Lambda_N \\ |\hat{\imath} - \hat{\kappa}| = 1}} \int_{S_N} \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\kappa}}, \xi)} - \sqrt{f(\eta, \xi)} \right] \sqrt{f(\eta, \xi)} \, d\nu_N(\eta, \xi).$$

We use then the change of variable $\tilde{\eta} := \eta^{\hat{\imath}, \hat{\kappa}}$ in the second integral. Since the measure ν_N is a Bernoulli product measure with constant parameter γ, $\nu_N(\eta, \xi) = \nu_N(\eta^{\hat{\imath}, \hat{\kappa}}, \xi)$, for any $(\eta, \xi) \in S_N$. We thus have

$$-\mathbb{D}_N^{\mathrm{field}}(\sqrt{f}, \nu_N) = \frac{d}{4} \sum_{\substack{\hat{\imath}, \hat{\kappa} \in \Lambda_N \\ |\hat{\imath} - \hat{\kappa}| = 1}} \int_{S_N} \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\kappa}}, \xi)} - \sqrt{f(\eta, \xi)} \right] \sqrt{f(\eta, \xi)} \, d\nu_N(\eta, \xi)$$

$$+ \frac{d}{4} \sum_{\substack{\hat{\imath}, \hat{\kappa} \in \Lambda_N \\ |\hat{\imath} - \hat{\kappa}| = 1}} \int_{S_N} \left[\sqrt{f(\eta, \xi)} - \sqrt{f(\eta^{\hat{\imath}, \hat{\kappa}}, \xi)} \right] \sqrt{f(\eta^{\hat{\imath}, \hat{\kappa}}, \xi)} \, d\nu_N(\eta, \xi)$$

from which follows (A.5). $\qquad \Box$

Proof of Lemma A.1, (D2). Follow the same method as in the proof of (D1). $\qquad \Box$

Proof of Lemma A.1, (D3). For clarity, let us introduce the notations

$$F := \sqrt{f(\eta, \xi)} \qquad \text{and} \qquad F^i := \sqrt{f(\eta^i, \xi)}. \qquad (A.11)$$

By writing $\mathbb{D}_N^{\mathrm{field}}$ from (A.2) and (2.4), we get

$$-D_N^{\mathrm{Rob}}(\sqrt{f}, \nu_N) = \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \int_{S_N} (\eta(i) - \xi(i))^2 (F^i - F) F \, d\nu_N(\eta, \xi)$$

$$+ \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \int_{\substack{S_N \\ \eta(i) = 0}} (\eta(i) - \xi(i))^2 (F^i - F) F \, d\nu_N(\eta, \xi)$$

$$+ \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \int_{\substack{S_N \\ \eta(i) = 1}} (\eta(i) - \xi(i))^2 (F^i - F) F \, d\nu_N(\eta, \xi).$$

We then use the change of variable $\widetilde{\eta} := \eta^i$ in the integral of the second and third line so that we have

$$\nu_N(\eta, \xi) = \nu_N(\widetilde{\eta}, \xi) \times \begin{cases} (1-\gamma)/\gamma & \text{if } \eta(i) = 0, \\ \gamma/(1-\gamma) & \text{if } \eta(i) = 1, \end{cases} = \nu_N(\widetilde{\eta}, \xi) \times \begin{cases} 1 + \frac{1-2\gamma}{\gamma} & \text{if } \eta(i) = 0, \\ 1 + \frac{2\gamma-1}{1-\gamma} & \text{if } \eta(i) = 1. \end{cases}$$

This results in

$$
\begin{aligned}
-D_N^{\mathrm{Rob}}(\sqrt{f}, \nu_N) = {} & \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \int_{S_N} (\eta(i) - \xi(i))^2 (F^i - F) F \, d\nu_N(\eta, \xi) \\
& + \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \int_{S_N} (1 - \eta(i) - \xi(i))^2 (F - F^i) F^i \, d\nu_N(\eta, \xi) \\
& + \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \frac{1-2\gamma}{\gamma} \int_{\substack{S_N \\ \widetilde{\eta}(i)=1}} (1 - \widetilde{\eta}(i) - \xi(i))^2 (F - F^i) F^i \, d\nu_N(\widetilde{\eta}, \xi) \\
& + \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \frac{2\gamma-1}{1-\gamma} \int_{\substack{S_N \\ \widetilde{\eta}(i)=0}} (1 - \widetilde{\eta}(i) - \xi(i))^2 (F - F^i) F^i \, d\nu_N(\widetilde{\eta}, \xi).
\end{aligned}
$$

Now observe that $(1 - \eta(i) - \xi(i))^2 = (\eta(i) - \xi(i))^2 + (1 - 2\eta(i))(1 - 2\xi(i))$ and plug this into the second line above. This directly yields (A.7) with

$$
\begin{aligned}
\mathcal{E}_N(\alpha, \gamma, f) = {} & \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \int_{S_N} (1 - 2\eta(i))(1 - 2\xi(i))(F - F^i) F^i \, d\nu_N(\eta, \xi) \\
& + \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \frac{1-2\gamma}{\gamma} \int_{\substack{S_N \\ \eta(i)=1}} (1 - \eta(i) - \xi(i))^2 (F - F^i) F^i \, d\nu_N(\eta, \xi) \\
& + \frac{\alpha}{2} \sum_{i \in \Gamma_N^{\mathrm{low}}} \frac{2\gamma-1}{1-\gamma} \int_{\substack{S_N \\ \eta(i)=0}} (1 - \eta(i) - \xi(i))^2 (F - F^i) F^i \, d\nu_N(\eta, \xi),
\end{aligned}
$$

and the control of $\mathcal{E}_N$ (A.10) arises from the Cauchy-Schwarz inequality and the fact that f is a density with respect to the Bernoulli product measure ν_N. $\qquad\square$

Proof of Lemma A.1, (D4). Follow the same method as in the proof of (D3). $\qquad\square$

Proof of Lemma A.1, (D5). As in the proof of (D3), we employ the notations F and F^i defined in (A.11). By writing D_N^{up} from (A.2) and (2.6), we get

$$
\begin{aligned}
-D_N^{\mathrm{up}}(\sqrt{f}, \nu_N) = {} & b \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \eta(i)=0}} (F^i F - \tfrac{1}{2} F^2) \, d\nu_N(\eta, \xi) - b \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \eta(i)=0}} \tfrac{1}{2} F^2 \, d\nu_N(\eta, \xi) \\
& + (1 - b) \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \eta(i)=1}} (F^i F - \tfrac{1}{2} F^2) \, d\nu_N(\eta, \xi) - (1 - b) \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \eta(i)=1}} \tfrac{1}{2} F^2 \, d\nu_N(\eta, \xi).
\end{aligned}
$$

We then use the change of variable $\widetilde{\eta} := \eta^i$ in the second and fourth integrals above so that we have

$$\nu_N(\widetilde{\eta}, \xi) = \nu_N(\eta, \xi) \times \begin{cases} \gamma/(1-\gamma) & \text{if } \eta(i) = 0, \\ (1-\gamma)/\gamma & \text{if } \eta(i) = 1, \end{cases}$$

resulting in

$$
\begin{aligned}
-D_N^{\mathrm{up}}(\sqrt{f}, \nu_N) = {} & b \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \eta(i)=0}} (F^i F - \tfrac{1}{2} F^2) \, d\nu_N(\eta, \xi) - \tfrac{(1-\gamma)b}{\gamma} \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \widetilde{\eta}(i)=1}} \tfrac{1}{2} (F^i)^2 \, d\nu_N(\widetilde{\eta}, \xi) \\
& + (1 - b) \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \eta(i)=1}} (F^i F - \tfrac{1}{2} F^2) \, d\nu_N(\eta, \xi) - \tfrac{\gamma(1-b)}{1-\gamma} \sum_{i \in \Gamma_N^{\mathrm{up}}} \int_{\substack{S_N \\ \widetilde{\eta}(i)=0}} \tfrac{1}{2} (F^i)^2 \, d\nu_N(\widetilde{\eta}, \xi).
\end{aligned}
$$

By choosing $\gamma = b$, we get

$$
\begin{aligned}
-D_N^{\text{up}}(\sqrt{f}, \nu_N) &= -\frac{b}{2} \sum_{i \in \Gamma_N^{\text{up}}} \int_{\substack{S_N \\ \eta(i)=0}} \left(F^2 - 2F^i F + (F^i)^2 \right) d\nu_N(\eta, \xi) \\
&\quad - \frac{1-b}{2} \sum_{i \in \Gamma_N^{\text{up}}} \int_{\substack{S_N \\ \eta(i)=1}} \left(F^2 - 2F^i F + (F^i)^2 \right) d\nu_N(\eta, \xi) \\
&= -\frac{1}{2} \sum_{i \in \Gamma_N^{\text{up}}} \int_{S_N} \left(b(1 - \eta(i)) + (1-b)\eta(i) \right) \left(F^i - F \right)^2 d\nu_N(\eta, \xi),
\end{aligned}
$$

that is (A.9). $\qquad\square$

We also need some useful inequalities that are gathered in the following lemma.

Lemma A.2 (Useful inequalities)

(I1) *For any $a > 0$, any $p \in \mathbb{N}^*$, and any sequence of positive numbers (a_N) and (b_N), we have*

$$
\limsup_{N \to \infty} \frac{1}{aN^p} \log(a_N + b_N) \leq \max \left(\limsup_{N \to \infty} \frac{1}{aN^p} \log(a_N), \limsup_{N \to \infty} \frac{1}{aN^p} \log(b_N) \right).
$$

(I2) *For any $z \in \mathbb{R}$ we have $e^{|z|} \leq e^z + e^{-z}$.*

(I3) *For any $X, Y \in \mathbb{R}$ and any $B > 0$, we have $XY \leq \frac{1}{2B} X^2 + \frac{B}{2} Y^2$.*

(I4) *For any $X, Y \in \mathbb{R}$, we have $\frac{(X+Y)^2}{2} \leq X^2 + Y^2$.*

A.2   Proof of the upper Replacement lemma

Proof of Lemma 4.4 (Replacement at the upper boundary). Consider the term under the integral in (4.7), namely

$$
A_{N,\varepsilon}(G(s, \tfrac{\hat{\imath}}{N}), \eta_s) := \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Gamma_N^{\text{up}}} G(s, \tfrac{\hat{\imath}}{N}) \left[\eta_s^{\varepsilon N}(\hat{\imath}) - \eta_s(\hat{\imath}) \right]. \tag{A.12}
$$

We develop $\eta_s^{\varepsilon N}(\hat{\imath})$ in (A.12) with (4.5) and express that $\hat{\imath} + \hat{\kappa}$, with $\hat{\kappa} = (k, \ell - N + 1)$, browses $\Lambda_{\hat{\imath}}^{\varepsilon N}$. This yields

$$
A_{N,\varepsilon}(G(s, \tfrac{\hat{\imath}}{N}), \eta_s) = \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell = \lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{\hat{\imath} \in \Gamma_N^{\text{up}}} G(s, \tfrac{\hat{\imath}}{N}) \left[\eta_s(\hat{\imath} + \hat{\kappa}) - \eta_s(\hat{\imath}) \right]. \tag{A.13}
$$

In (A.13), focus on

$$
\sum_{\hat{\imath} \in \Gamma_N^{\text{up}}} G(s, \tfrac{\hat{\imath}}{N}) \eta_s(\hat{\imath} + \hat{\kappa}) = \sum_{\hat{\imath} \in \Gamma_N^{\text{up}}} \left[G(s, \tfrac{\hat{\imath}}{N}) - G(s, \tfrac{\hat{\imath}+\hat{\kappa}}{N}) \right] \eta_s(\hat{\imath} + \hat{\kappa}) + \sum_{\hat{\imath} \in \Gamma_N^{\text{up}}} G(s, \tfrac{\hat{\imath}+\hat{\kappa}}{N}) \eta_s(\hat{\imath} + \hat{\kappa}). \tag{A.14}
$$

Since $G \in \mathcal{C}^{1,2}([0,T] \times \overline{\Lambda})$, the Mean Value Theorem allows to control, in the first sum of (A.14),

$$|G(s, \tfrac{\hat{\imath}}{N}) - G(s, \tfrac{\hat{\imath}+\hat{\kappa}}{N})| \leq p\varepsilon \sup_{s \in [0,T]} \|\nabla G(s, \cdot)\|_{L^{\infty}(\Lambda)}. \qquad (A.15)$$

For the second sum in (A.14), notice that we have, thanks to the periodicity of the torus $\mathbb{T}_N$ and the Mean Value Theorem,

$$
\begin{aligned}
\sum_{\hat{\imath} \in \Gamma_N^{\mathrm{up}}} G(s, \tfrac{\hat{\imath}+\hat{\kappa}}{N})\eta_s(\hat{\imath}+\hat{\kappa}) &= \sum_{i \in \mathbb{T}_N^{p-1}} G(s, \tfrac{i+k}{N}, \tfrac{\ell}{N})\eta_s(i+k, \ell) \\
&= \sum_{i \in \mathbb{T}_N^{p-1}} G(s, \tfrac{i}{N}, \tfrac{\ell}{N})\eta_s(i, \ell) \\
&= \sum_{i \in \mathbb{T}_N^{p-1}} \left[G(s, \tfrac{i}{N}, \tfrac{N-1}{N}) + \frac{N-1-\ell}{N} C_1(G, N) \right] \eta_s(i, \ell), \quad (A.16)
\end{aligned}
$$

where there is $y \in (0,1)$ such that

$$|C_1(G, N)| = |\partial_y G(s, \tfrac{i}{N}, y)| \leq \sup_{s \in [0,T]} \|\nabla G(s, \cdot)\|_{L^{\infty}(\Lambda)}. \qquad (A.17)$$

Now using (A.14) and (A.16) into the expression of $A_{N,\varepsilon}$ in (A.13), we obtain

$$
A_{N,\varepsilon}(G(s, \tfrac{\hat{\imath}}{N}), \eta_s)
$$

$$
= \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{\hat{\imath} \in \Gamma_N^{\mathrm{up}}} \left[G(s, \tfrac{\hat{\imath}}{N}) - G(s, \tfrac{\hat{\imath}+\hat{\kappa}}{N}) \right] \eta_s(\hat{\imath}+\hat{\kappa}) \qquad (A.18)
$$

$$
+ \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i \in \mathbb{T}_N^{p-1}} \frac{N-1-\ell}{N} C_1(G, N)\eta_s(i, \ell) \qquad (A.19)
$$

$$
+ \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i \in \mathbb{T}_N^{p-1}} G(s, \tfrac{i}{N}, \tfrac{N-1}{N}) \left[\eta_s(i, \ell) - \eta_s(i, N-1) \right]. \qquad (A.20)
$$

Therefore we have

$$
\mathbb{E}_N^{\mu_N} \left[\left| \int_0^T \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Gamma_N^{\mathrm{up}}} G(s, \tfrac{\hat{\imath}}{N}) \left[\eta_s^{\varepsilon N}(\hat{\imath}) - \eta_s(\hat{\imath}) \right] ds \right| \right]
$$

$$
\leq \mathbb{E}_N^{\mu_N} \left[\left| \int_0^T (A.18)\, ds \right| \right] + \mathbb{E}_N^{\mu_N} \left[\left| \int_0^T (A.19)\, ds \right| \right] + \mathbb{E}_N^{\mu_N} \left[\left| \int_0^T (A.20)\, ds \right| \right]. \qquad (A.21)
$$

In (A.21), the vanishing of the two first expectations is a straight consequence of the upper bounds (A.15) and (A.17). Indeed, for the first expectation in (A.21), we have

$$
\mathbb{E}_N^{\mu_N}\left[\left|\int_0^T (A.18)\, ds\right|\right]
$$

$$
= \mathbb{E}_N^{\mu_N}\left[\left|\int_0^T \sum_{k\in[-\lceil\varepsilon N\rceil,\lceil\varepsilon N\rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N\rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{\hat{\imath}\in\Gamma_N^{\mathrm{up}}} \left[G(s,\tfrac{\hat{\imath}}{N}) - G(s,\tfrac{\hat{\imath}+\hat{\kappa}}{N})\right]\eta_s(\hat{\imath}+\hat{\kappa})\, ds\right|\right]
$$

$$
\leq \int_0^T \sum_{k\in[-\lceil\varepsilon N\rceil,\lceil\varepsilon N\rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N\rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{\hat{\imath}\in\Gamma_N^{\mathrm{up}}} \underbrace{\left|G(s,\tfrac{\hat{\imath}}{N}) - G(s,\tfrac{\hat{\imath}+\hat{\kappa}}{N})\right|}_{\text{control this with (A.15)}}\, ds
$$

$$
\leq Tp\varepsilon \sup_{s\in[0,T]} \|\nabla G(s,\cdot)\|_{L^\infty(\Lambda)} \times \underbrace{\sum_{k\in[-\lceil\varepsilon N\rceil,\lceil\varepsilon N\rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N\rceil}^{N-1} c_{N,\varepsilon}}_{=1} \times \underbrace{\frac{1}{N^{p-1}}\sum_{\hat{\imath}\in\Gamma_N^{\mathrm{up}}}1}_{=1}
$$

$$
= Tp\varepsilon \sup_{s\in[0,T]} \|\nabla G(s,\cdot)\|_{L^\infty(\Lambda)}
$$

that vanishes as $\varepsilon\to 0^+$. Similarly,

$$
\mathbb{E}_N^{\mu_N}\left[\left|\int_0^T (A.19)\, ds\right|\right]
$$

$$
= \mathbb{E}_N^{\mu_N}\left[\left|\int_0^T \sum_{k\in[-\lceil\varepsilon N\rceil,\lceil\varepsilon N\rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N\rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i\in\mathbb{T}_N^{p-1}} \overbrace{\frac{N-1-\ell}{N}}^{\leq\varepsilon} C_1(G,N)\eta_s(i,\ell)\, ds\right|\right]
$$

$$
\leq T\varepsilon \sum_{k\in[-\lceil\varepsilon N\rceil,\lceil\varepsilon N\rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N\rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{\hat{\imath}\in\Gamma_N^{\mathrm{up}}} \underbrace{|C_1(G,N)|}_{\text{control this with (A.17)}}
$$

$$
\leq T\varepsilon \sup_{s\in[0,T]} \|\nabla G(s,\cdot)\|_{L^\infty(\Lambda)}
$$

that also vanishes as $\varepsilon\to 0^+$. The last expectation outlined in (A.21), namely

$$
\mathbb{E}_N^{\mu_N}\left[\left|\int_0^T (A.20)\, ds\right|\right]
$$

$$
= \mathbb{E}_N^{\mu_N}\left[\left|\int_0^T \sum_{k\in[-\lceil\varepsilon N\rceil,\lceil\varepsilon N\rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N\rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i\in\mathbb{T}_N^{p-1}} G(s,\tfrac{i}{N},\tfrac{N-1}{N})\left[\eta_s(i,\ell) - \eta_s(i,N-1)\right]\, ds\right|\right]
$$

$$
\tag{A.22}
$$

captures key information about the y-direction of η_s in the region $\Lambda_{\hat{\imath}}^{\varepsilon N}$. The vanishing of this expectation is actually the core of this proof.

Given any fixed $a>0$ (which will eventually be increased to $+\infty$), we use the entropy inequality (see [34, Appendix 1, Section 8]) on (A.22). This yields

$$
\mathbb{E}_N^{\mu_N}\left[\left|\int_0^T (A.20)\, ds\right|\right] \leq \frac{1}{aN^p}\log\left[\mathbb{E}_N^{\nu_N}\left[\exp\left(aN^p\left|\int_0^T (A.20)\, ds\right|\right)\right]\right] + \frac{1}{aN^p}\mathcal{H}(\mu_N|\nu_N). \tag{A.23}
$$

Thanks to the control we have on $\mathcal{H}(\mu_N | \nu_N)$, as outlined in (A.1), the second term in (A.23) is bounded by C_0/a that vanishes when $a \to \infty$. Therefore, to conclude the proof, it is enough to show that, for any $a > 0$,

$$\lim_{\varepsilon \to 0^+} \lim_{N \to \infty} \frac{1}{aN^p} \log \left[\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \left| \int_0^T (A.20) \, ds \right| \right) \right] \right] = 0.$$

Now, we remark that we can get rid of the absolute value into the exponential of this last expression. Indeed, by combining (I2) and (I1) in Lemma A.2 with

$$z = aN^p \int_0^T (A.20) \, ds, \qquad a_n = \mathbb{E}_N^{\nu_N}(e^z), \qquad \text{and} \qquad b_n = \mathbb{E}_N^{\nu_N}(e^{-z}),$$

we have

$$\limsup_{N \to \infty} \frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N}(e^{|z|}) \right) \leq \max \left[\limsup_{N \to \infty} \frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N}(e^z) \right), \limsup_{N \to \infty} \frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N}(e^{-z}) \right) \right],$$

and therefore (up to take $-G$ instead of G) we only have to prove, when first $N \to \infty$, and then $\varepsilon \to 0^+$, the vanishing of

$$\frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N}(e^z) \right) = \frac{1}{aN^p} \log \left[\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \int_0^T (A.20) \, ds \right) \right] \right].$$

We use now the Feynman-Kac's inequality — see [4, Lemma 7.3 in Appendix] — with the operator $\mathcal{L}_N + aN^p V(s, \cdot)$ where

$$V(s, \cdot) : \eta \mapsto \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell = \lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i \in \mathbb{T}_N^{p-1}} G(s, \tfrac{i}{N}, \tfrac{N-1}{N}) \left[\eta(i, \ell) - \eta(i, N-1) \right]. \quad (A.24)$$

With the variational formula (Rayleigh quotient) for the largest (principal) eigenvalue of the operator $\mathcal{L}_N + aN^p V(s, \cdot)$, we are led to

$$\frac{1}{aN^p} \log \left[\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \int_0^T (A.20) \, ds \right) \right] \right]$$

$$\leq \int_0^T \sup_{\substack{f \text{ density} \\ \text{with respect} \\ \text{to } \nu_N}} \left\{ \int_{S_N} V(s, \eta) \, f(\eta, \xi) \, d\nu_N(\eta, \xi) + \frac{1}{aN^p} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_N} \right\} ds. \quad (A.25)$$

Now focus on the integral term into the supremum in (A.25). By using a telescopic sum to write the differences $\eta(i, \ell) - \eta(i, N-1)$ in $V(s, \eta)$ as defined in (A.24), we get

$$\int_{S_N} V(s, \eta) \, f(\eta, \xi) \, d\nu_N(\eta, \xi)$$

$$= \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell = \lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i \in \mathbb{T}_N^{p-1}} G(s, \tfrac{i}{N}, \tfrac{N-1}{N})$$

$$\times \sum_{m=\ell}^{N-2} \int_{S_N} \left[\eta(i, m) - \eta(i, m+1) \right] f(\eta, \xi) \, d\nu_N(\eta, \xi). \quad (A.26)$$

In (A.26), we focus on

$$\int_{S_N} [\eta(i,m) - \eta(i,m+1)] \, f(\eta,\xi) \, d\nu_N(\eta,\xi)$$
$$= \int_{S_N} \eta(i,m) \, f(\eta,\xi) \, d\nu_N(\eta,\xi) - \int_{S_N} \eta(i,m+1) \, f(\eta,\xi) \, d\nu_N(\eta,\xi). \qquad \text{(A.27)}$$

In the first integral in (A.27), we perform the change of variable

$$\eta \mapsto \eta^{(i,m),(i,m+1)} =: \eta^{\hat{\kappa}_0,\hat{\kappa}_1}.$$

Since $\nu_N(\eta,\xi) = \nu_N(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)$ for any $(\eta,\xi) \in S_N$, this gives

$$\int_{S_N} [\eta(i,m) - \eta(i,m+1)] \, f(\eta,\xi) \, d\nu_N(\eta,\xi)$$
$$= \int_{S_N} \eta(\hat{\kappa}_1) \left[f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi) - f(\eta,\xi) \right] d\nu_N(\eta,\xi)$$
$$= \int_{S_N} \eta(\hat{\kappa}_1) \underbrace{\left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} - \sqrt{f(\eta,\xi)} \right] \left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} + \sqrt{f(\eta,\xi)} \right]}_{\text{use (I3) at this point}} d\nu_N(\eta,\xi).$$

$$\leq \frac{B}{2} \int_{S_N} \eta(\hat{\kappa}_1) \left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} - \sqrt{f(\eta,\xi)} \right]^2 d\nu_N(\eta,\xi) \qquad \text{(A.28)}$$
$$+ \frac{1}{2B} \int_{S_N} \eta(\hat{\kappa}_1) \left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} + \sqrt{f(\eta,\xi)} \right]^2 d\nu_N(\eta,\xi), \qquad \text{(A.29)}$$

where $B > 0$ is to be determined later in this proof. In (A.29) we now use (I4) and the facts that f is a density with respect to ν_N, $0 \leq \eta(\hat{\kappa}_1) \leq 1$, and $\nu_N(\eta,\xi) = \nu_N(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)$ for any $(\eta,\xi) \in S_N$, to write

$$\int_{S_N} [\eta(i,m) - \eta(i,m+1)] \, f(\eta,\xi) \, d\nu_N(\eta,\xi)$$
$$\leq \frac{B}{2} \int_{S_N} \eta(\hat{\kappa}_1) \left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} - \sqrt{f(\eta,\xi)} \right]^2 d\nu_N(\eta,\xi) + \frac{2}{B}. \qquad \text{(A.30)}$$

Incorporating (A.30) into (A.26) yields

$$\int_{S_N} V(s,\eta) \, f(\eta,\xi) \, d\nu_N(\eta,\xi)$$
$$\leq \sum_{k \in [-\lceil \varepsilon N \rceil, \lceil \varepsilon N \rceil]^{p-1}} \sum_{\ell=\lceil N-1-\varepsilon N \rceil}^{N-1} \frac{c_{N,\varepsilon}}{N^{p-1}} \sum_{i \in \mathbb{T}_N^{p-1}} G(s, \tfrac{i}{N}, \tfrac{N-1}{N})$$
$$\times \sum_{m=\ell}^{N-2} \left[\frac{B}{2} \int_{S_N} \eta(\hat{\kappa}_1) \left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} - \sqrt{f(\eta,\xi)} \right]^2 d\nu_N(\eta,\xi) + \frac{2}{B} \right]$$
$$\leq \|G(s,\cdot)\|_{L^\infty(\Lambda)} \left[\frac{B}{2N^{p-1}} \sum_{i \in \mathbb{T}_N^{p-1}} \sum_{m=0}^{N-2} \int_{S_N} \left[\sqrt{f(\eta^{\hat{\kappa}_0,\hat{\kappa}_1},\xi)} - \sqrt{f(\eta,\xi)} \right]^2 d\nu_N(\eta,\xi) + \frac{2\varepsilon N}{B} \right]$$
$$\text{(A.31)}$$

where we used that $\sum_k \sum_\ell c_{N,\varepsilon} = 1$, $\frac{1}{N^{p-1}} \sum_i 1 = 1$, and

$$N - 1 - \ell \leq \varepsilon N \qquad \text{for all } \ell \in [\![\lceil N-1-\varepsilon N \rceil; N-1]\!].$$

Now observe that we can make the Dirichlet form $\mathbb{D}_N^{\text{field}}$ appear in (A.31) thanks to (D1). This results in

$$\int_{S_N} V(s,\eta)\, f(\eta,\xi)\, d\nu_N(\eta,\xi) \leq \|G(s,\cdot)\|_{L^\infty(\Lambda)} \left[\frac{2B}{dN^{p-1}}\, \mathbb{D}_N^{\text{field}}(\sqrt{f},\nu_N) + \frac{2\varepsilon N}{B} \right]. \quad (A.32)$$

By gathering (A.23), (A.25) and (A.32), we obtain

$$
\begin{aligned}
&\mathbb{E}_N^{\mu_N}\!\left[\left| \int_0^T (A.20)\, ds \right| \right] \\
&\leq \int_0^T \sup_{\substack{f \text{ density} \\ \text{with respect} \\ \text{to } \nu_N}} \left\{ \|G(s,\cdot)\|_{L^\infty(\Lambda)} \left[\frac{2B}{dN^{p-1}}\, \mathbb{D}_N^{\text{field}}(\sqrt{f},\nu_N) + \frac{2\varepsilon N}{B} \right] + \frac{1}{aN^p} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_N} \right\} ds + \frac{C_0}{a} \\
&\leq \int_0^T \left[\sup_{\substack{f \text{ density} \\ \text{with respect} \\ \text{to } \nu_N}} \left\{ \left[\frac{2B\|G(s,\cdot)\|_{L^\infty(\Lambda)}}{dN^{p-1}} - \frac{1}{aN^{p-2}} \right] \mathbb{D}_N^{\text{field}}(\sqrt{f},\nu_N) \right\} + \frac{2\varepsilon N \|G(s,\cdot)\|_{L^\infty(\Lambda)}}{B} \right] ds + \frac{C_0 + 2c(b)\alpha}{a},
\end{aligned}
$$
$$(A.33)$$

where we used the whole Lemma A.1 (with $\gamma = b$ in (A.10)) and the fact that $-\frac{1}{2}\mathbb{I}_N^{\text{road}}$, $-\frac{1}{2}I_N^{\text{up}}$, $-\frac{1}{2}I_N^{\text{reac}}$ and $-\frac{1}{2}I_N^{\text{Rob}}$ are nonpositive to provide the last inequality. At this point, we make the choice $B := dN/(2a\|G(s,\cdot)\|_{L^\infty(\Lambda)})$ to cancel the supremum in (A.33), so that we are left with

$$\mathbb{E}_N^{\mu_N}\!\left[\left| \int_0^T (A.20)\, ds \right| \right] \leq \frac{4T\varepsilon a}{d} \sup_{s\in[0,T]} \|G(s,\cdot)\|_{L^\infty(\Lambda)}^2 + \frac{C_0 + 2c(b)\alpha}{a}. \quad (A.34)$$

By letting $\varepsilon \to 0^+$ and then $a \to \infty$, we finally get the vanishing of $\mathbb{E}_N^{\mu_N}[|\int_0^T (A.20)\, ds|]$ which, combined with those of $\mathbb{E}_N^{\mu_N}[|\int_0^T (A.18)\, ds|]$ and $\mathbb{E}_N^{\mu_N}[|\int_0^T (A.19)\, ds|]$ in (A.21), completes this proof. $\qquad\square$

A.3 Proof of the energy estimate

Proof of Lemma 4.3 (Energy estimate). For $q \in [\![1;p]\!]$, $G \in \mathcal{C}_c^{0,2}([0,T]\times\overline{\Lambda})$ and $v \in L^2(0,T;L^2(\Lambda))$, let us write the quantity below the temporal integral in (4.3) as

$$J_G(s) = J_G(s,v,q) := \left\langle v(s), \partial_{e_q} G(s) \right\rangle_\Lambda - \frac{1}{2}\|G(s)\|_{L^2(\Lambda)}^2.$$

Now consider a sequence $(G_n)_{n\in\mathbb{N}}$ dense in $\mathcal{C}_c^{0,2}([0,T]\times\overline{\Lambda})$, and observe that it is sufficient to show that there is a constant $C > 0$ such that for any $n_0 \in \mathbb{N}$, we have

$$\mathbb{E}_\infty\!\left[\max_{0\leq n\leq n_0} \left\{ \int_0^T J_{G_n}(s)\, ds \right\} \right] < C, \quad (A.35)$$

where $\mathbb{E}_\infty$ denotes the expectation with respect to $\mathbb{Q}_\infty$. Since the maps

$$(\pi_N(t))_{t\in[0,T]} \mapsto \int_0^T \left(\langle \pi_N^{\text{field}}(s), \partial_{e_q} G_n(s) \rangle - \frac{1}{2}\|G_n(s)\|_{L^2(\Lambda)}^2 \right) ds$$

are continuous with respect to the Skorokhod topology, and since the probability $\mathbb{Q}_\infty$ is in the weak closure of $(\mathbb{Q}_N^{\mu_N})_{N\geq 2}$, then the expectation in (A.35) can be recast as

$$\lim_{N\to\infty} \mathbb{E}_N^{\mu_N}\!\left[\max_{0\leq n\leq n_0} \left\{ \int_0^T \left(\langle \pi_N^{\text{field}}(s), \partial_{e_q} G_n(s) \rangle - \frac{1}{2}\|G_n(s)\|_{L^2(\Lambda)}^2 \right) ds \right\} \right],$$

that is, by explaining the empirical measure $\pi_N^{\text{field}}(s)$,

$$\lim_{N \to \infty} \mathbb{E}_N^{\mu_N} \left[\max_{0 \leq n \leq n_0} \left\{ \int_0^T \Big(\underbrace{\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} \partial_{e_q} G_n(s, \tfrac{\hat{\imath}}{N}) \times \eta_s(\hat{\imath}) - \frac{1}{2} \|G_n(s)\|_{L^2(\Lambda)}^2}_{=: J_{G_n}^N(s,\eta,q) = J_{G_n}^N(s).} \Big) ds \right\} \right] \qquad \text{(A.36)}$$

For a fixed $a > 0$, we use the entropy inequality (see [34, Appendix 1, Section 8]) to bound the expectation in (A.36) with

$$\mathbb{E}_N^{\mu_N} \left[\max_{0 \leq n \leq n_0} \left\{ \int_0^T J_{G_n}^N(s) \, ds \right\} \right]$$

$$\leq \frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \times \max_{0 \leq n \leq n_0} \left\{ \int_0^T J_{G_n}^N(s) \, ds \right\} \right) \right] \right) + \frac{1}{aN^p} \mathcal{H}(\mu_N | \nu_N). \qquad \text{(A.37)}$$

Thanks to the control we have on $\mathcal{H}(\mu_N | \nu_N)$, as outlined in (A.1), the second term in (A.37) is bounded by C_0/a and does not pose a significant issue as long as a remains far from 0. Now focusing on the first term in (A.37), notice that

$$\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \times \max_{0 \leq n \leq n_0} \left\{ \int_0^T J_{G_n}^N(s) \, ds \right\} \right) \right] \leq \mathbb{E}_N^{\nu_N} \left[\sum_{n=0}^{n_0} \exp \left(aN^p \int_0^T J_{G_n}^N(s) \, ds \right) \right]$$

$$= \sum_{n=0}^{n_0} \mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \int_0^T J_{G_n}^N(s) \, ds \right) \right].$$

This control allows to bootstrap (I1) (cf. Lemma A.2), providing

$$\limsup_{N \to \infty} \left\{ \frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \times \max_{0 \leq n \leq n_0} \left\{ \int_0^T J_{G_n}^N(s) \, ds \right\} \right) \right] \right) \right\}$$

$$\leq \max_{0 \leq n \leq n_0} \limsup_{N \to \infty} \left\{ \frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \times \int_0^T J_{G_n}^N(s) \, ds \right) \right] \right) \right\}. \qquad \text{(A.38)}$$

Working on the term between the bracket in (A.38), we use the Feynman-Kac's inequality — see [4] (Lemma 7.3 in Appendix) — with the operator $\mathcal{L}_N + aN^p J_{G_n}^N(s, \cdot, q)$. By using the variational formula (Rayleigh quotient) for the largest (principal) eigenvalue of this operator, we are led to

$$\frac{1}{aN^p} \log \left(\mathbb{E}_N^{\nu_N} \left[\exp \left(aN^p \times \int_0^T J_{G_n}^N(s) \, ds \right) \right] \right)$$

$$\leq \int_0^T \sup_{\substack{f \text{ density} \\ \text{with respect} \\ \text{to } \nu_N}} \left\{ \int_{S_N} \Big(\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} \eta(\hat{\imath}) \partial_{e_q} G_n(s, \tfrac{\hat{\imath}}{N}) \Big) f(\eta, \xi) \, d\nu_N(\eta, \xi) + \frac{1}{aN^p} \langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_N} \right\} ds \qquad \text{(A.39)}$$

$$- \frac{1}{2} \int_0^T \|G_n(s)\|_{L^2(\Lambda)}^2 \, ds.$$

Considering the integral term into the supremum in (A.39), the regularity of G_n enables to write

$$\frac{1}{N^p} \partial_{e_q} G_n(s, \tfrac{\hat{\imath}}{N}) = \frac{1}{N^{p-1}} \Big(G_n(s, \tfrac{\hat{\imath}+e_q}{N}) - G_n(s, \tfrac{\hat{\imath}}{N}) + o(1/N) \Big),$$

so that

$$\int_{S_N} \left(\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} \eta(\hat{\imath}) \partial_{e_q} G_n(s, \tfrac{\hat{\imath}}{N}) \right) f(\eta, \xi) \, d\nu_N(\eta, \xi)$$

$$= \int_{S_N} \left(\frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \eta(\hat{\imath}) \left(G_n(s, \tfrac{\hat{\imath}+e_q}{N}) - G_n(s, \tfrac{\hat{\imath}}{N}) \right) \right) f(\eta, \xi) \, d\nu_N(\eta, \xi) + o_N(1).$$

Using then a summation by parts (mind the compact support of G) and the change of variable $\eta \mapsto \eta^{\hat{\imath}, \hat{\imath}-e_q}$, from the fact that $\nu_N(\eta, \xi) = \nu_N(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)$ for any $(\eta, \xi) \in S_N$, we obtain

$$\int_{S_N} \left(\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} \eta(\hat{\imath}) \partial_{e_q} G_n(s, \tfrac{\hat{\imath}}{N}) \right) f(\eta, \xi) \, d\nu_N(\eta, \xi) - o_N(1)$$

$$= \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \int_{S_N} \eta(\hat{\imath}) \, G_n(s, \tfrac{\hat{\imath}}{N}) \left(f(\eta, \xi) - f(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi) \right) \, d\nu_N(\eta, \xi)$$

$$= -\frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \int_{S_N} \eta(\hat{\imath}) \underbrace{G_n(s, \tfrac{\hat{\imath}}{N}) \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)} - \sqrt{f(\eta, \xi)} \right] \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)} + \sqrt{f(\eta, \xi)} \right]}_{\text{use (I3) at this point}} d\nu_N(\eta, \xi)$$

$$\leq \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \frac{B}{2} \int_{S_N} \eta(\hat{\imath}) \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi) \tag{A.40}$$

$$+ \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \frac{1}{2B} \left(G_n(s, \tfrac{\hat{\imath}}{N}) \right)^2 \int_{S_N} \eta(\hat{\imath}) \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)} + \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi), \tag{A.41}$$

where $B > 0$ is to be determined later in this proof. In (A.41) we use now (I4) and the facts that f is a density with respect to ν_N, $0 \leq \eta(\hat{\imath}) \leq 1$, and $\nu_N(\eta, \xi) = \nu_N(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)$ for any $(\eta, \xi) \in S_N$, to write

$$\int_{S_N} \left(\frac{1}{N^p} \sum_{\hat{\imath} \in \Lambda_N} \eta(\hat{\imath}) \partial_{e_q} G_n(s, \tfrac{\hat{\imath}}{N}) \right) f(\eta, \xi) \, d\nu_N(\eta, \xi) - o_N(1)$$

$$\leq \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \frac{B}{2} \int_{S_N} \eta(\hat{\imath}) \left[\sqrt{f(\eta^{\hat{\imath}, \hat{\imath}-e_q}, \xi)} - \sqrt{f(\eta, \xi)} \right]^2 d\nu_N(\eta, \xi) + \frac{1}{N^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \frac{2}{B} \left(G_n(s, \tfrac{\hat{\imath}}{N}) \right)^2$$

$$\leq \frac{2B}{dN^{p-1}} \mathbb{D}_N^{\text{field}}(\sqrt{f}, \nu_N) + \frac{2}{BN^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \left(G_n(s, \tfrac{\hat{\imath}}{N}) \right)^2, \tag{A.42}$$

where we used (D1) to provide (A.42). By gathering (A.37), (A.39) and (A.42), we obtain

$$
\mathbb{E}_N^{\mu_N}\left[\max_{0 \le n \le n_0} \left\{ \int_0^T J_{G_n}^N(s)\, ds \right\} \right]
$$

$$
\le \int_0^T \sup_{\substack{f \text{ density} \\ \text{with respect} \\ \text{to } \nu_N}} \left\{ \frac{2B}{dN^{p-1}} \mathbb{D}_N^{\text{field}}(\sqrt{f}, \nu_N) + \frac{1}{aN^p}\langle \mathcal{L}_N \sqrt{f}, \sqrt{f} \rangle_{\nu_N} \right\} ds
$$

$$
+ \int_0^T \left(\frac{2}{BN^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \left(G_n(s, \tfrac{\hat{\imath}}{N}) \right)^2 - \frac{1}{2}\|G_n(s)\|_{L^2(\Lambda)}^2 \right) ds + \frac{C_0}{a} + o_N(1)
$$

$$
\le \int_0^T \sup_{\substack{f \text{ density} \\ \text{with respect} \\ \text{to } \nu_N}} \left\{ \left[\frac{2B}{dN^{p-1}} - \frac{1}{aN^{p-2}} \right] \mathbb{D}_N^{\text{field}}(\sqrt{f}, \nu_N) \right\} ds \tag{A.43}
$$

$$
+ \int_0^T \left(\frac{2}{BN^{p-1}} \sum_{\hat{\imath} \in \Lambda_N} \left(G_n(s, \tfrac{\hat{\imath}}{N}) \right)^2 - \frac{1}{2}\|G_n(s)\|_{L^2(\Lambda)}^2 \right) ds + \frac{C_0 + c(b)\alpha}{a} + o_N(1),
$$

where we used Lemma A.1 (with $\gamma = b$ in (A.10)) to provide the last inequality. At this point, we make the choice $B := dN/2a$ to cancel the supremum in (A.43), so that we are left with

$$
\mathbb{E}_N^{\mu_N}\left[\max_{0 \le n \le n_0} \left\{ \int_0^T J_{G_n}^N(s)\, ds \right\} \right]
$$

$$
\le \int_0^T \left(\frac{4a}{dN^p} \sum_{\hat{\imath} \in \Lambda_N} \left(G_n(s, \tfrac{\hat{\imath}}{N}) \right)^2 - \frac{1}{2}\|G_n(s)\|_{L^2(\Lambda)}^2 \right) ds + \frac{C_0 + c(b)\alpha}{a} + o_N(1). \tag{A.44}
$$

We take now $a := d/8$ to face the norm with the Riemann sum in (A.44), this results in

$$
\mathbb{E}_N^{\mu_N}\left[\max_{0 \le n \le n_0} \left\{ \int_0^T J_{G_n}^N(s)\, ds \right\} \right] \le \frac{8C_0 + 8c(b)\alpha}{d} + o_N(1).
$$

It remains to let $N \to \infty$ to eventually reach (A.35) with $C := (8C_0 + 8c(b)\alpha)/d$. This completes the proof. $\qquad\square$

Table of Notations

Notation	Description
p	Dimension of the field (the road is $(p-1)$-dimensional)
$\mathbb{T}$	One-dimensional torus $\mathbb{R}/\mathbb{Z}$
$\mathbb{T}^{p-1}$	Macroscopic road
Λ	Macroscopic field $\mathbb{T}^{p-1} \times (0,1)$
$\overline{\Lambda}$	Closure of Λ
Γ	Macroscopic frontier of the field $\partial\Lambda = \mathbb{T}^{p-1} \times \{0,1\}$

Γ^{up}	Macroscopic upper boundary of the field
Γ^{low}	Macroscopic lower boundary of the field
N	Size of the microscopic particle system
$\mathbb{T}_N$	One-dimensional discrete torus $\mathbb{Z}/N\mathbb{Z}$
$\mathbb{T}_N^{p-1}$	Microscopic road
Λ_N	Microscopic field $\mathbb{T}_N \times [\![1; N-1]\!]$
Γ_N	Microscopic frontier of the field $\partial\Lambda_N = \mathbb{T}_N \times \{0, 1\}$
Γ_N^{up}	Microscopic upper boundary of the field
Γ_N^{low}	Microscopic lower boundary of the field
$\hat{\imath} = (i, j)$	Microscopic point on Λ_N ($i \in \mathbb{T}_N$ and $j \in [\![1; N-1]\!]$)
$\hat{\kappa} = (k, \ell)$	Alternative microscopic point on Λ_N if needed
$\hat{x} = (x, y)$	Macroscopic point on Λ ($x \in \mathbb{T}$ and $y \in (0, 1)$)
$\hat{z} = (z, \omega)$	Alternative macroscopic point on Λ if needed
e_q	q^{th} canonical vector of $\mathbb{R}^p$ ($1 \le q \le p$)
$\tilde{e}_q$	q^{th} canonical vector of $\mathbb{R}^{p-1}$ ($1 \le q \le p-1$)
(η, ξ)	State of the system (η for the field and ξ for the road)
S_N^{field}	State space for the field $\{0, 1\}^{\Lambda_N} \ni \eta$
S_N^{road}	State space for the road $\{0, 1\}^{\Gamma_N^{\mathrm{low}}} \ni \xi$
S_N	Whole state space $S_N^{\mathrm{field}} \times S_N^{\mathrm{road}}$
$\pi_N^{\mathrm{field}}(\eta)$	Empirical measure on S_N^{field} associated with η
$\pi_N^{\mathrm{road}}(\xi)$	Empirical measure on S_N^{road} associated with ξ
$\pi_N(\eta, \xi)$	Empirical measure on S_N associated with (η, ξ)
$\mathcal{M}^{\mathrm{field}}$	Set of positive measures on S_N^{field} bounded by 1
$\mathcal{M}^{\mathrm{road}}$	Set of positive measures on S_N^{road} bounded by 1
$\mathcal{M}$	Cartesian product $\mathcal{M}^{\mathrm{field}} \times \mathcal{M}^{\mathrm{road}}$
T	Time horizon
d, D	Diffusion coefficients
α	Exchange coefficient
b	Birth rate at the upper boundary
γ	Parameter of the Bernoulli product measures
$\mathbb{P}_N^{\mu_N}$	Probability measure on S_N induced by μ_N and $(\eta_t, \xi_t)_{t \in [0, T]}$
$\mathbb{Q}_N^{\mu_N}$	Probability measure on $\mathcal{M}$ induced by $\mathbb{P}_N^{\mu_N}$ and π_N
$\mathbb{Q}_\infty$	A point in the closure of $(\mathbb{Q}_N^{\mu_N})_{N \ge 2}$
$\mathbb{E}_N^{\bullet}$	Expectation with respect to $\mathbb{P}_N^{\bullet}$
$\mathbb{E}_N^{\mu_N}$	Expectation with respect to $\mathbb{P}_N^{\mu_N}$

$\mathbb{E}_N^{\nu_N}$	Expectation with respect to $\mathbb{P}_N^{\nu_N}$
$\mathbb{E}_\infty$	Expectation with respect to $\mathbb{Q}_\infty$
Δ	Laplacian operator for the field
Δ_x	Laplacian operator for the road
Δ_x^N	Discrete Laplacian operator for the road
∂_{yy}^N	Discrete Laplacian operator in the y-direction
∇	Gradient operator for the field
∇_x	Gradient operator for the road
Tr	Trace operator
G	Test functions for the field
H	Test functions for the road
$\mathbb{L}_N^{\text{field}}$	Bulk field part of the generator
$\mathbb{L}_N^{\text{road}}$	Bulk road part of the generator
L_N^{Rob}	Robin exchange part of the generator
L_N^{reac}	Reaction exchange part of the generator
L_N^{up}	Upper reservoir part of the generator
$\mathcal{D}_N$	Dirichlet form
$\mathbb{D}_N^{\text{field}}$	Bulk field part of the Dirichlet form
$\mathbb{D}_N^{\text{road}}$	Bulk road part of the Dirichlet form
D_N^{Rob}	Robin exchange part of the Dirichlet form
D_N^{reac}	Reaction exchange part of the Dirichlet form
D_N^{up}	Upper reservoir part of the Dirichlet form
$\mathcal{H}(\mu|\nu)$	Relative entropy of μ with respect to ν
$\mathscr{M}_N(t)$	The Martingale
$\mathscr{M}_{N,G}^{\text{field}}(t)$	Field part of the Martingale $\mathscr{M}_N(t)$
$\mathscr{M}_{N,H}^{\text{road}}(t)$	Road part of the Martingale $\mathscr{M}_N(t)$
$\mathscr{B}_{N,G}^{\text{field}}(t)$	"B"-part of the martingale $\mathscr{N}_{N,G}^{\text{field}}(t)$
$\mathscr{B}_{N,H}^{\text{road}}(t)$	"B"-part of the martingale $\mathscr{N}_{N,H}^{\text{road}}(t)$
$\mathscr{S}_{\text{W1}}$	Set of measures whose densities satisfy (W1)
$\mathscr{S}_{\text{W2}}$	Set of measures whose densities satisfy (W2)
$\mathscr{W}_{u,v,(G,H)}(T)$	Functional of the weak formulation $\mathscr{W}_{u,v,(G,H)} = \mathscr{W}_{v,G}^{\text{field}} + \mathscr{W}_{u,H}^{\text{road}}$
$\mathscr{W}_{v,G}^{\text{field}}(T)$	Functional of the weak formulation in the field
$\mathscr{W}_{u,H}^{\text{road}}(T)$	Functional of the weak formulation on the road
$\mathbb{U}_\varepsilon^{\text{up}}$	Upper unit approximation
$\mathbb{U}_\varepsilon^{\text{low}}$	Upper unit approximation

References

[1] E. AFFILI, *A Fisher-KPP model with a fast diffusion line in periodic media*, arXiv:2009.14760, (2020).

[2] M. ALFARO AND C. CHAINAIS-HILLAIRET, *Long time behavior of the field-road diffusion model: An entropy method and a finite volume scheme*, arXiv:2309.16242, (2023).

[3] M. ALFARO, R. DUCASSE, AND S. TRÉTON, *The field-road diffusion model: Fundamental solution and asymptotic behavior*, Journal of Differential Equations, 367 (2023), pp. 332–365.

[4] R. BALDASSO, O. MENEZES, A. NEUMANN, AND R. R. SOUZA, *Exclusion Process with Slow Boundary*, Journal of Statistical Physics, 167 (2017), pp. 1112–1142.

[5] H. BERESTYCKI, A.-C. COULON, J.-M. ROQUEJOFFRE, AND L. ROSSI, *Speed-up of reaction-diffusion fronts by a line of fast diffusion*, Séminaire Laurent Schwartz — EDP et applications, (2013/2014), pp. 1–25.

[6] ——, *The effect of a line with nonlocal diffusion on Fisher-KPP propagation*, Mathematical Models and Methods in Applied Sciences, 25 (2015), pp. 2519–2562.

[7] H. BERESTYCKI, R. DUCASSE, AND L. ROSSI, *Generalized principal eigenvalues for heterogeneous road–field systems*, Communications in Contemporary Mathematics, 22 (2020), p. 1950013.

[8] ——, *Influence of a road on a population in an ecological niche facing climate change*, Journal of Mathematical Biology, 81 (2020), pp. 1059–1097.

[9] H. BERESTYCKI, J.-M. ROQUEJOFFRE, AND L. ROSSI, *Fisher–KPP propagation in the presence of a line: Further effects*, Nonlinearity, 26 (2013), pp. 2623–2640.

[10] ——, *The influence of a line with fast diffusion on Fisher-KPP propagation*, Journal of Mathematical Biology, 66 (2013), pp. 743–766.

[11] ——, *The shape of expansion induced by a line with fast diffusion in Fisher-KPP equations*, Communications in Mathematical Physics, 343 (2016), pp. 207–232.

[12] ——, *Travelling waves, spreading and extinction for Fisher–KPP propagation driven by a line with fast diffusion*, Nonlinear Analysis, 137 (2016), pp. 171–189.

[13] C. BERNARDIN, P. GONCALVES, AND B. O. JIMENEZ, *Slow to fast infinitely extended reservoirs for the symmetric exclusion process with long jumps*, Markov Processes and Related Fields, 25 (2019), pp. 217–274.

[14] P. BILLINGSLEY, *Convergence of Probability Measures*, Wiley Series in Probability and Statistics, John Wiley & Sons, Inc., New York, 2nd ed., 2013.

[15] T. BODINEAU, B. DERRIDA, AND J. L. LEBOWITZ, *A Diffusive System Driven by a Battery or by a Smoothly Varying Field*, Journal of Statistical Physics, 140 (2010), pp. 648–675.

[16] B. BOGOSEL, T. GILETTI, AND A. TELLINI, *Propagation for KPP bulk-surface systems in a general cylindrical domain*, Nonlinear Analysis, 213 (2021), p. 112528.

[17] H. BREZIS, *Analyse Fonctionnelle*, Masson, théorie et applications ed., 1983.

[18] D. CUSSEDDU, L. EDELSTEIN-KESHET, J. A. MACKENZIE, S. PORTET, AND A. MADZVAMUSE, *A coupled bulk-surface model for cell polarisation*, Journal of Theoretical Biology, 481 (2019), pp. 119–135.

[19] A. DE MASI, E. PRESUTTI, D. TSAGKAROGIANNIS, AND M. E. VARES, *Current Reservoirs in the Simple Exclusion Process*, Journal of Statistical Physics, 144 (2011), pp. 1151–1170.

[20] ——, *Non-equilibrium Stationary States in the Symmetric Simple Exclusion with Births and Deaths*, Journal of Statistical Physics, 147 (2012), pp. 519–528.

[21] ——, *Truncated correlations in the stirring process with births and deaths*, Electronic Journal of Probability, 17 (2012), pp. 1–35.

[22] R. DUCASSE, *Influence of the geometry on a field-road model: The case of a conical field*, Journal of the London Mathematical Society, 97 (2018), pp. 441–469.

[23] H. EGGER, K. FELLNER, J. F. PIETSCHMANN, AND B. Q. TANG, *Analysis and numerical solution of coupled volume-surface reaction-diffusion systems with application to cell biology*, Applied Mathematics and Computation, 336 (2018), pp. 351–367.

[24] L. C. EVANS, *Partial Differential Equations*, vol. 19 of Graduate Studies in Mathematics, American Mathematical Society, Providence (Rhode Island), 2nd ed., 2010.

[25] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, Textbooks in Mathematics, CRC Press, Boca Raton (Florida), 2nd ed., 2015.

[26] K. FELLNER, E. LATOS, AND B. Q. TANG, *Well-posedness and exponential equilibration of a volume-surface reaction–diffusion system with nonlinear boundary coupling*, Annales de l'Institut Henri Poincaré C, Analyse non linéaire, 35 (2018), pp. 643–673.

[27] T. FRANCO, P. GONÇALVES, AND A. NEUMANN, *Hydrodynamical behavior of symmetric exclusion with slow bonds*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 49 (2013), pp. 402–427.

[28] ——, *Phase transition in equilibrium fluctuations of symmetric slowed exclusion*, Stochastic Processes and their Applications, 123 (2013), pp. 4156–4185.

[29] ——, *Phase transition of a heat equation with Robin's boundary conditions and exclusion process*, Transactions of the American Mathematical Society, 367 (2015), pp. 6131–6158.

[30] T. Franco and M. Tavares, *Hydrodynamic Limit for the SSEP with a Slow Membrane*, Journal of Statistical Physics, 175 (2019), pp. 233–268.

[31] M. Gatto, E. Bertuzzo, L. Mari, S. Miccoli, L. Carraro, R. Casagrandi, and A. Rinaldo, *Spread and dynamics of the COVID-19 epidemic in Italy: Effects of emergency containment measures*, Proceedings of the National Academy of Sciences, 117 (2020), pp. 10484–10491.

[32] M.-H. Giga, Y. Giga, and J. Saal, *Nonlinear Partial Differential Equations: Asymptotic Behavior of Solutions and Self-Similar Solutions*, Springer Science & Business Media, 2010.

[33] T. Giletti, L. Monsaingeon, and M. Zhou, *A KPP road–field system with spatially periodic exchange terms*, Nonlinear Analysis, 128 (2015), pp. 273–302.

[34] C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems*, vol. 320 of Comprehensive Studies in Mathematics, Springer, New York, 1999.

[35] K. Kuoch, M. Mourragui, and E. Saada, *A boundary driven generalized contact process with exchange of particles: Hydrodynamics in infinite volume*, Stochastic Processes and their Applications, 127 (2017), pp. 135–178.

[36] C. Landim, A. Milanés, and S. Olla, *Stationary and Nonequilibrium Fluctuations in Boundary Driven Exclusion Processes*, Markov Processes and Related Fields, 14 (2008), pp. 165–184.

[37] C. Landim and K. Tsunoda, *Hydrostatics and dynamical large deviations for a reaction-diffusion model*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 54 (2018), pp. 51–74.

[38] C. Landim and S. Velasco, *Dynamic and static large deviations of a one dimensional SSEP in weak contact with reservoirs*, Aug. 2023.

[39] T. M. Liggett, *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, vol. 324 of Grundlehren Der Mathematischen Wissenschaften, Springer, Berlin, Heidelberg, 1999.

[40] ——, *Interacting Particle Systems*, vol. 2 of Classics in Mathematics, Springer, Berlin, Heidelberg, 2005.

[41] H. W. McKenzie, E. H. Merrill, R. J. Spiteri, and M. A. Lewis, *How linear features alter predator movement and the functional response*, Interface Focus, 2 (2012), pp. 205–216.

[42] M. Mourragui, E. Saada, and S. Velasco, *Hydrodynamic and hydrostatic limit for a generalized contact process with mixed boundary conditions*, Electronic Journal of Probability, 28 (2023), pp. 1–44.

[43] A. Pauthier, *The influence of nonlocal exchange terms on Fisher–KPP propagation driven by a line of fast diffusion*, Communications in Mathematical Sciences, 14 (2015), pp. 535–570.

[44] ——, *Road-field reaction-diffusion system: A new threshold for long range exchanges*, arXiv:1504.05437, (2015).

[45] ——, *Uniform dynamics for Fisher-KPP propagation driven by a line of fast diffusion under a singular limit*, Nonlinearity, 28 (2015), pp. 3891–3920.

[46] C. Robinet, C.-E. Imbert, J. Rousselet, D. Sauvard, J. Garcia, F. Goussard, and A. Roques, *Human-mediated long-distance jumps of the pine processionary moth in Europe*, Biological Invasions, 14 (2012), pp. 1557–1569.

[47] L. Rossi, A. Tellini, and E. Valdinoci, *The effect on Fisher-KPP propagation in a cylinder with fast diffusion on the boundary*, SIAM Journal on Mathematical Analysis, 49 (2017), pp. 4595–4624.

[48] W. Rudin, *Real and complex analysis*, McGraw-Hill Book Co., New York, third ed., 1987.

[49] B. V. Schmid, U. Büntgen, W. R. Easterday, C. Ginzler, L. Walløe, B. Bramanti, and N. C. Stenseth, *Climate-driven introduction of the Black Death and successive plague reintroductions into Europe*, Proceedings of the National Academy of Sciences, 112 (2015), pp. 3020–3025.

[50] F. Spitzer, *Interaction of Markov processes*, Advances in Mathematics, 5 (1970), pp. 246–290.

[51] H. Spohn, *Long range correlations for stochastic lattice gases in a non-equilibrium steady state*, Journal of Physics A: Mathematical and General, 16 (1983), p. 4275.

[52] A. Tellini, *Propagation speed in a strip bounded by a line with different diffusion*, Journal of Differential Equations, 260 (2016), pp. 5956–5986.

[53] S. Tréton, *Analyse de Dynamiques d'Échanges Microscopiques et Macroscopiques pour l'Écologie et l'Épidémiologie*, PhD thesis, University of Rouen, 2024.

[54] M. Zhang, *Spreading speeds and pulsating fronts for a field-road model in a spatially periodic habitat*, Journal of Differential Equations, 304 (2021), pp. 191–228.