Master's thesis in mathematics

Reaction-Diffusion Equations on a Field-Road space

Submitted by
Samuel TRÉTON

Under the supervision of
Matthieu ALFARO

UNIVERSITY OF ROUEN NORMANDY, FRANCE

February – July 2021

Remerciments

Je tiens à adresser mes meilleurs remerciements à mon directeur de stage Matthieu Alfaro tout d'abord pour m'avoir accordé sa confiance en me choisissant pour stagiaire, puis pour avoir m'avoir conseillé et guidé dans mon travail avec une remarquable performance autant sur le plan logistique que mathématique. Je le remercie également de m'avoir laissé une grande marge de manoeuvre dans mes actions, notamment dans l'écriture de ce mémoire, ainsi que de la pédagogie dont il a fait preuve afin d'améliorer ma compréhension. Je tire un grande satisfaction du travail que nous avons pu faire ensemble et j'espère vivement avoir le plaisir de poursuivre les recherches que nous avons entreprises.

Je remercie le Laboratoire de Mathématique Raphaël Salem de m'avoir accordé une bourse pour ce stage, je souhaite aussi remercier Adel Blouza pour les conseils qu'il m'a donnés pour mes simulations numériques, je remercie également pour leur soutien et leur heureuse compagnie mes camarades de master (Abdeltif, Arthur, Léopold), mes aînés doctorants (Anas, Cécile, Cyril, Gauthier, Mirella, Rheadel, Victor) mais aussi mes chers voisins de bureau Séverin et Anouar ! Enfin mes derniers remerciement vont pour ma fiancée Myriam en qui je puise toute cette énergie qui m'anime et me pousse à donner le meilleur de moi-même.

Table of content

Introduction 1

Part I Reaction-Diffusion Equations in $\mathbb{R}^N$ 3

- I.1 Reaction ... 3
- I.2 Diffusion in $\mathbb{R}^N$ ... 11
- I.3 Reaction-Diffusion in $\mathbb{R}^N$ ... 18
- I.4 Fisher-KPP equation ... 24
- I.5 Weak Allee effect ... 29

Part II Fisher-KPP Reaction-Diffusion Equations on the Field-Road space $\mathbb{R}^2_+$ 38

- II.1 Presentation of the Field-Road model ... 40
- II.2 Population mass conservation ... 42
- II.3 A few words about the μ parameter ... 44
- II.4 Cauchy problem: Existence, uniqueness, CP ... 45
- II.5 Long time behaviour ... 48
- II.6 Exponential super-solutions ... 58
- II.7 Asymptotic spreading speed C_* ... 64
- II.8 C_* behaviour as D tends to $+\infty$ ... 74

Part III Research on the Field-Road space $\mathbb{R}^N_+$ 81

- III.1 Heat kernel in the half-space $\mathbb{R}_+$ ... 82
- III.2 Heat kernel in the half-space $\mathbb{R}^N_+$ ... 94

Scripts for numerical implementation **107**

Toolbox **117**

Bibliography **123**

Notations **124**

Index **129**

Introduction

In that Master's thesis, we shall deal with the so-called Reaction-Diffusion Equations. These are being used in a large range of fields such as chemical reactions studies, flame propagations, evolution of some heated environment, neutron-scattering theory, genetics, epidemiology... The discusion here points towards the population dynamics framework; we therefore shall consider some groups of individuals evolving from an initial condition over time and space due to their borns, deaths, and displacements. As an applied mathematical domain, our aim is to present some concrete results in order to build some explicit links between this report and the study of ecology. That's why we mainly shall wonder whether the species under consideration will be extinct in long time or persist and invade the space. This report is divided in three main parts:

- In first part, we discuss on Reaction-Diffusion Equations in the whole $\mathbb{R}^N$ space. It is an occasion to present to the reader a fiew kinds of reaction functions used in population dynamics and to show some classical results such as the Hair Trigger Effect theorem for Fisher-KPP reactions or the Aronson and Weinberger's one.

- Second part consists in an presentation of the Field-Road model proposed in the article of Berestycki *et al.* in the paper [4] published in 2018. That pattern allows to induce some "fast diffusion channels" which modelises some faster displacements of individuals along a certain axe (the Road) compared to the rest of the domain (the Field). A fiew examples – like the one of the canadian wolfs – that justify the need to developp such a pattern are given in the begining of the part. One adapts then, following the paper of Berestycki *et al.*, the classical Hair Trigger Effect theorem for Fisher-KPP reaction to that Field-Road model and one finally deals with some asymptotic spreading speed results.

- The third and last part contains some elements of research on the Berestycki *et al.* Field-Road model: by introducing an Allee effect on the reaction term that acts on the Field, our goal is to achieve an equivalent theorem of the Aronson and Weinberger's one in the Field-Road case. To do that, we have to follow the thread of the proof of that theorem in the classical $\mathbb{R}^N$ space which finally brings us to consider the heat equation for that Field-Road system. As far we know, no such development has been done in the litterature yet. To approach that issue one has firstly simplified the problem by only considering the heat equation on an half space provided with some homogenous Robin boundary conditions; such conditions constitute actually a key aspect of that Field-Road model which is then paramount to sharply understand before any further considerations.

Finally, we end this report with some additionnal content, that is

- some pieces of code in Scilab and FreeFem for numerical implementation,
- a toolbox for technical results,
- the bibliography,
- the correspondances to the notations used in the whole document.

If it may help him/her, the reader will also find an index of the principal notions in the two last pages.

Wish you a nice read!
ST.

PART I

Reaction-Diffusion Equations in $\mathbb{R}^N$

I.1 Reaction

We shall deal in this section with some non-spatial models. Considering a given population evolving over time and let $u(t) \in \mathbb{R}_+$ denote the size of that population ([a]) at time $t \geq 0$. We suppose here the temporal evolution of the amount $u(t)$ is deterministically lead by a *reaction function* $f : \mathbb{R} \to \mathbb{R}$ *via* the following autonomous ODE:

$$u'(t) = f(u(t)).$$

Coupling the latter with some initial datum

$$u(0) = u_0 \geq 0,$$

we obtain a Cauchy problem.

The aim of this section is to recall some general and useful results about ODE theory and to introduce the most classical examples used for the reaction function f in population dynamics.

I.1.1 ODE: general and useful results

Let thus consider the following Cauchy problem:

$$\begin{cases} u'(t) = f(u(t)) & t \in (0;\infty) \\ u(0) = u_0, & \end{cases} \qquad (\mathbf{I.1})$$

where $f \in \mathcal{C}^0(\mathbb{R}, \mathbb{R})$; we start by recalling the notion of solution for this problem.

[a] $u(t)$ may also be the population density, assuming the latter is homogeneously distributed in space.

Definition 1 (Cauchy problem solution)
One calls a *solution* of (I.1) each couple *"validity-interval/function"* $([0;T), u)$ such that

- $T \in \mathbb{R}_+^* \cup \{+\infty\}$, and
- $u \in \mathcal{C}^1([0;T), \mathbb{R})$ satisfies
  - first line of (I.1) for all $t \in (0;T)$,
  - second line of (I.1).

A solution of (I.1) is said *global* if $T = +\infty$.

We now announce the Cauchy-Lipschitz theorem which is the starting point of the ODE theory. It gives us, under relatively weak assumptions on the function f, an existence and uniqueness result for (I.1).

Theorem 2 (Cauchy-Lipschitz)
Consider the Cauchy problem (I.1).

$\boxed{\mathbf{1}}$ If $f : \mathbb{R} \to \mathbb{R}$ is

- continuous on $\mathbb{R}$,
- *locally* Lipschitz continuous on $\mathbb{R}$,

then (I.1) admit a unique maximal solution $([0;T), u)$ in the sense that whether $([0;T_1), u_1)$ is another solution of (I.1) with the same initial datum, we necessarily get $T_1 \leq T$.

$\boxed{\mathbf{2}}$ Assume moreover we know *a priori* that

- there exists $I \subset \mathbb{R}$ such that $u(t) \in I$ for all $t \geq 0$,
- f is *globally* Lipschitz continuous on I,

then the solution $([0;T), u)$ given by $\boxed{\mathbf{1}}$ is actually global.

Remarks.

- Two different solutions of $u' = f(u)$ never "cross".
- If u and v are solutions of (I.1) with respective initial datums $u_0 < v_0$, then $u(t) < v(t)$ for all $t \geq 0$.

Proposition 3 (Sufficient conditions for Lipschitz continuity)

- Let $I \subseteq \mathbb{R}$, assume $f \in \mathcal{C}^1(I, \mathbb{R})$, then f is *locally* Lipschitz continuous on I.
- If f' is moreover bounded on I ([a]), then f is *globally* Lipschitz continuous on I.

[a] It is the case as soon as I is compact.

Hypothesis 1: We suppose from here that $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$.

Definition 4 (Equilibrium point)
A point $u_E \in \mathbb{R}$ is said an *equilibrium point* for the ODE $u' = f(u)$ if $f(u_E) = 0$.

Remarks.

- If u_E is an equilibrium point for $u' = f(u)$, then $u \equiv u_E$ is the unique global solution starting from u_E.
- If $u_{E_-} < u_{E_+}$ are two equilibrium points for $u' = f(u)$, then for all $u_{E_-} < u_0 < u_{E_+}$, the solution u starting from u_0 is global and verifies $u_{E_-} < u(t) < u_{E_+}$ for all $t \geq 0$.
- In population dynamics, it seems realistic to choose the reaction function f such that 0 is an equilibrium; from one part because the initial datum $u_0 = 0$ naturally leads to $u \equiv 0$ and from another part because it creates without artifice a barrier between positive and negatives values of u ($u < 0$ is devoid of physical meaning).
- If all equilibriums points of $u' = f(u)$ are isolated, then, thanks to $f \in \mathcal{C}^1$, the solutions are either constant (if the initial datum is an equilibrium) or strictly monotonous (if the initial datum is between two successive equilibriums).

According to the last remark, one distinguish two kinds of equilibrium points according their stability:

- *asymptotically stable equilibriums*: there exists a neighbourhood of the considering equilibrium point such that each solution starting from this neighbourhood is global and converges to the equilibrium as t tends to $+\infty$.
- *not asymptotically stable equilibriums*: all equilibrium points which are not asymptotically stable.

Theorem 5 (Linearization around equilibrium points)
Assume $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$ and let $u_E \in \mathbb{R}$ be an equilibrium point for the ODE $u' = f(u)$.

- If $f'(u_E) < 0$ then u_E is *asymptotically stable.*
- If $f'(u_E) > 0$ then u_E is *not asymptotically stable.*

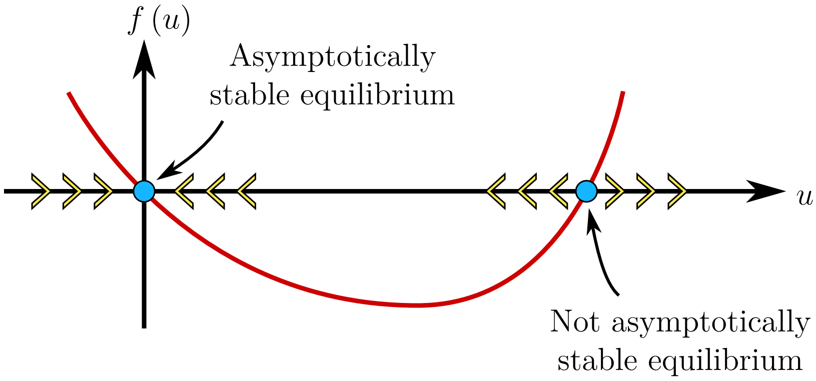


Figure F1 – *Illustration of two equilibrium points for $u' = f(u)$ represented in blue. The graph of f has been drown in red. Imagine an initial point u_0 on the u-axis, then this point moves over time, to the left if $f < 0$ and to the right if $f > 0$.*

Remark. Under the assumptions of theorem 5, the case $f'(u_E) = 0$ is a degenerate one and does not allow us any conclusion. To be convinced, see for example figure (*F2*) below.

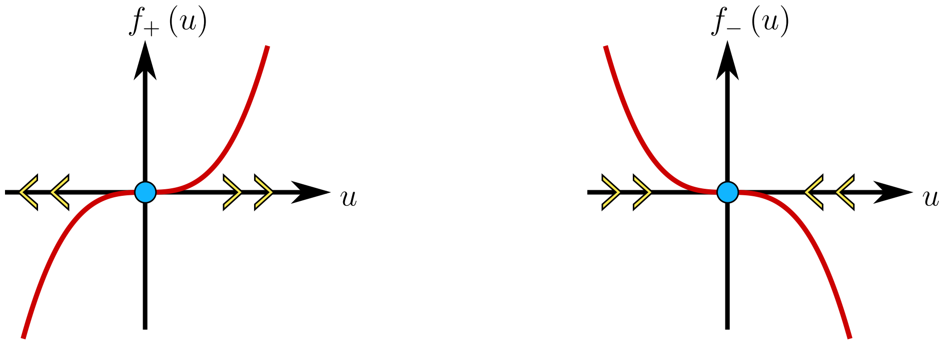


Figure F2 – *Two examples which do not fall within the scope of theorem 5. Let us take $f_\pm(x) = \pm x^3$, then $u_E = 0$ is an equilibrium point but $f'_\pm(0) = 0$. Therefore, no conclusion is possible: as one sees on the figure, u_E is asymptotically stable for $u' = f_-(u)$ but not for $u' = f_+(u)$,*

I.1.2 Classical examples for the reaction function f

We present here some examples of reaction functions used in population dynamics. For this purpose, we are especially based on the books of Roques [15] and Cantrell and Cosner [5]. Before going into details, we start by giving the definition of growth rate *per capita* which is an important notion to describe a model of reaction.

Definition 6 (Growth rate *per capita*)
One calls *growth rate per capita* of the reaction function f the amount

$$\tau_f : u \mapsto \begin{cases} f(u)/u & \text{if } u \neq 0 \\ f'(0) & \text{otherwise.} \end{cases}$$

Remarks.

- The growth rate *per capita* assess the average growth rate for a single individual.
- At first sight, it might be surprising to set $\tau_f(0) = f'(0)$; but it is actually not. Indeed, under the assumption $f(0) = 0$, we have

$$f'(0) = \lim_{u \to 0} \frac{f(0+u) - f(0)}{u} = \lim_{u \to 0} \tau_f(u),$$

  i.e. $f'(0)$ is none other than the extension by continuity of τ_f in $u = 0$.

Linear model

The first model we shall discuss on has been formulated by Malthus in 1798 who thought [10] that human population was exponentially increasing due to the fact that number of births and deaths was in proportional relation with population size. Thereby, the reaction function given by this model is defined by

$$f(u) = ru,$$

where the constant $r \in \mathbb{R}$ denotes the growth rate *per capita* τ_f whose sign set the growth $(r > 0)$ or the decay $(r < 0)$ of u. This model supposes natural resources ([b]) unlimited which may turn out to be unrealistic for large values of u. Indeed, we observe *a posteriori* that there exists an *intraspecific competition* for natural resources which strongly influence individual's births and deaths when u tends to be large. Thus, because of these interactions, number of births and deaths cannot stay linear with respect to u for high values of this one.

Logistic model

In order to take into consideration the *intraspecific competition* we noticed above, the mathematician Verhulst suggested around 1840 [16] that the growth rate *per capita* was a decreasing affine function – and not a constant function as it was supposed in the Malthus model. We thus pose

$$\tau_f(u) = r\left(1 - \frac{u}{K}\right), \qquad (r, K > 0).$$

In that way, the growth rate *per capita* is maximal for $u = 0$ and becomes all the more low as u becomes large due to competition between individuals. This leads to that expression for the reaction f:

$$f(u) = ru\left(1 - \frac{u}{K}\right).$$

[b] Available space, feed, etc.

We may notice that $u_E = K$ is an asymptotically stable equilibrium point for $u' = f(u)$; that value is called the *carrying capacity* and depends on the available natural resources and on the ability of individuals to share these resources.

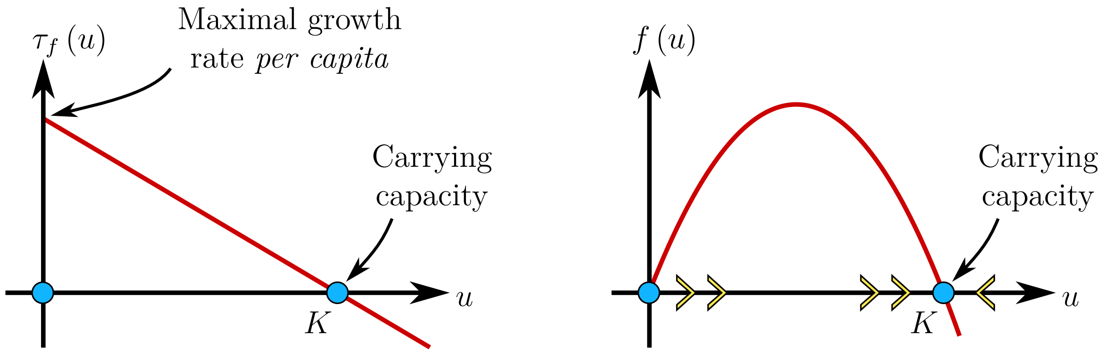


Figure F3 – *Logistic model : on the left, the growth rate per capita which is maximal when $u = 0$. On the right, the reaction function f.*

KPP hypothesis *versus* Allee effect

In fact, the logistic reaction fits into a more general group of models which stand out by satisfying the following hypothesis:

Definition 7 (KPP hypothesis)
We say that the reaction f satisfies the *KPP hypothesis* if the next inequality is verified:

$$f(u) \leq uf'(0) \qquad \forall u \geq 0.$$

Remark. The KPP hypothesis can also be defined as follows: f verifies the KPP hypothesis if and only if the growth rate *per capita* $\tau_f(u)$ is maximal when $u = 0$.

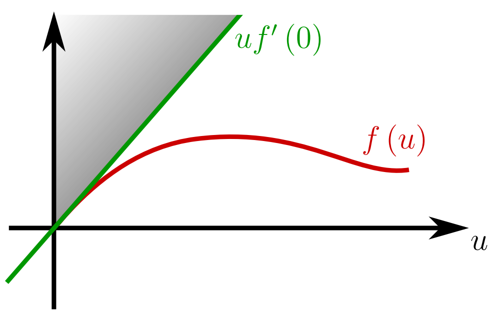


Figure F4 – *Illustration of KPP hypothesis: the graph of f cannot enter in the shaded area.*

With the intention of finding models even closer to reality, one can notice that the interactions between individuals do not only occur as intraspecific competition but also as mutual self-help whose absence might be prejudicial for the population development. These mutualist interactions or their absences (c) may lead to

- some hurdles to growth for small populations caused for example by
  - consanguinity,
  - difficulties in finding a sexual partner,
  - lower resistance to extreme weather phenomena,
- more facilities to growth for larger populations due for example to
  - genetic mixing,
  - cooperative hunt,
  - group of defence,
  - higher resistance to extreme weather phenomena.

This consideration brings us to touch on the notion of *Allee effect* that we define now.

Definition 8 (Allee effect)
One says that the reaction function f owns an *Allee effet* if its growth rate *per capita* $\tau_f(u)$ is not maximal for $u = 0$.

If f owns an Allee effect and satisfies moreover $\tau_f(0) < 0$ we talk about *strong* Allee effect; otherwise the Allee effect is qualified *weak*.

Remark. The both properties

"*to own an Allee effect*"

"*to satisfy the KPP hypothesis*"

are antagonistic each other.

Monostable degenerate model

The model we shall show here owns a weak Allee effect. Let us take the following reaction function:

$$f(u) = ru^{1+p}\left(1 - \frac{u}{K}\right), \qquad (r, p, K > 0).$$

We easily assess the growth rate *per capita* associated:

$$\tau_f(u) = ru^p\left(1 - \frac{u}{K}\right)$$

whom maxima is achieved for $u = Kp/(1+p) > 0$.

c Described for the first time by Allee in 1931 [1] and whom one finds a depth study in the Courchamp's book [6].

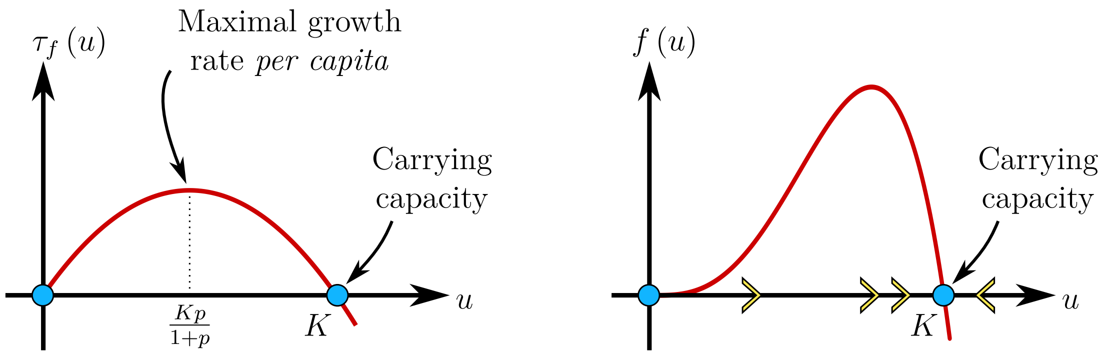


Figure F5 – *Monostable degenerate model (weak Allee effect) : on the left, the growth rate per capita which is not maximal when $u = 0$, i.e. there is an Allee effect; furthermore, $\tau_f(0) = 0$ is non-negative whence the Allee effect is a weak one. On the right, the reaction function f.*

Remark. The value p measures the intensity of the Allee effect:

- $p = 0$: no Allee effect (it is the logistic case),
- $p \ll 1$: less intense Allee effect,
- $p \gg 1$: more intense Allee effect.

Bistable model

The last model we present in this section is the bistable one. It is a classic example to illustrate some strong Allee effect. The form of the reaction function f is of this type:

$$f(u) = ru\left(1 - \frac{u}{K}\right)(u - \rho), \qquad (r, K > 0, \qquad 0 < \rho < K).$$

The parameter ρ [d] is a threshold demarcating extinction and persistence of the population in the following mean: consider the solution of the ODE $u' = f(u)$ starting from the initial datum $u_0 \geq 0$, then two cases are possible.

- If $0 \leq u_0 < \rho$, then u tend to 0 as t tends to $+\infty$, *i.e.* the population becomes extinct.
- If $u_0 \geq \rho$, then u does not tend to 0 as t tends to $+\infty$, *i.e.* the population persists.

The growth rate *per capita* is given by

$$\tau_f(u) = r\left(1 - \frac{u}{K}\right)(u - \rho)$$

which is maximal when $u = (K + \rho)/2 > 0$.

[d] Which is also a not asymptotically stable equilibrium point for $u' = f(u)$.

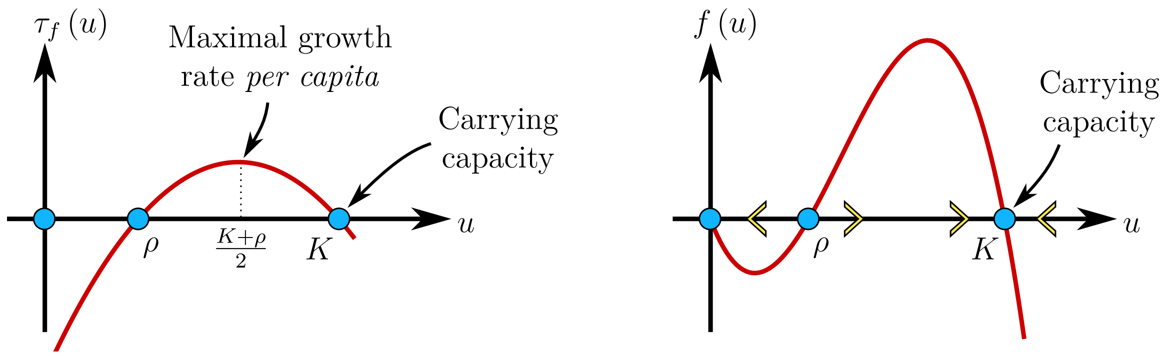


Figure F6 – *Bistable model (strong Allee effect) : on the left, the growth rate per capita which is not maximal when* $u = 0$*, i.e. there is an Allee effect; furthermore,* $\tau_f(0) < 0$ *whence the Allee effect is a strong one. On the right, the reaction function* f.

I.2 Diffusion in $\mathbb{R}^N$

Now that we have presented non-spatial reaction models which simulate the population size evolution, we approach the spatial problem of diffusion which modelize population spreading in space. Let Ω be an open set of $\mathbb{R}^N$ with $N \in \mathbb{N}^\star$, $X \in \Omega$ and $t \geq 0$; the amount $u(t, X)$ denotes from here the population density on place X at time t. We shall consider in this whole section the *diffusion equation* otherwise called *heat equation*

$$\partial_t u = d\Delta u,$$

whose we begin by motivate the use.

I.2.1 Diffusion equation obtaining *via* random walks

An intuitive way to introduce diffusion phenomenon in population dynamics is through random walks. We get place in the sequence in $\mathbb{R}^2$ but the entire reasoning is valid in $\mathbb{R}^N$. Let us consider a single individual moving every step of time $\delta_T > 0$ on a discrete grid dipped in $\mathbb{R}^2$ whom distance between each pair of adjacent points equals $\delta_S > 0$. From a position (x, y) on the space grid at discrete time $t \geq 0$, the individual owns two degrees of freedom to move the distance δ_S until the next instant $t + \delta_t$: up/down and left/right. The direction borrowed among the four possible is randomly chosen in an independent way of the previous moves and all the directions have the same probability to be taken – equals therefore $1/4$. The situation has been summarized on figure ($F7$) below.

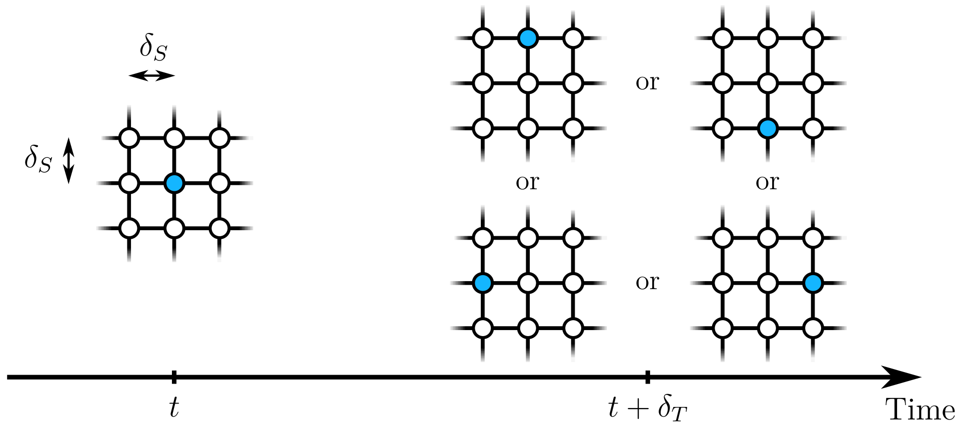


Figure F7 *– Sketch of the individual motion – represented in blue – during the random walk at times t and $t+\delta_T$. Starting from the state drawn in left, at time t, there are four equiprobable places where the individual may be at the next discrete time. Each possibility has thus a probability of 1/4.*

Let now (x,y) on the space grid and a discrete time $t \geq 0$. We note $p(t,x,y)$ the probability that the individual take place in (x,y) at time t. Thus, we have

$$p(t+\delta_T,x,y) = \frac{1}{4}\Big(p(t,x+\delta_S,y) + p(t,x-\delta_S,y) + p(t,x,y+\delta_S) + p(t,x,y-\delta_S)\Big)$$

whence

$$\frac{p(t+\delta_T,x,y)-p(t,x,y)}{\delta_T} = \frac{1}{4\delta_T}\Big(p(t,x+\delta_S,y) - 2p(t,x,y) + p(t,x-\delta_S,y) + p(t,x,y+\delta_S) - 2p(t,x,y) + p(t,x,y-\delta_S)\Big).$$

Let $d>0$, by assuming
$$\frac{\delta_S^2}{4\delta_T} = d \iff \frac{1}{4\delta_T} = \frac{d}{\delta_S^2},$$
we get

$$\frac{p(t+\delta_T,x,y)-p(t,x,y)}{\delta_T} = d\left(\frac{p(t,x+\delta_S,y) - 2p(t,x,y) + p(t,x-\delta_S,y)}{\delta_S^2} + \frac{p(t,x,y+\delta_S) - 2p(t,x,y) + p(t,x,y-\delta_S)}{\delta_S^2}\right).$$

Finally, one takes the limit [e] when δ_S and δ_T tend to 0 while preserving the ratio $d = \delta_S^2/(4\delta_T)$; we obtain the heat equation we sought

$$\partial_t u = d\Delta u.$$

[e] As it is done in the book of Okubo *et al.* [12], section 5.3 pages 133-134.

I.2.2 Diffusion equation with initial datum

Coming back to the N-dimensional case, we consider now the Cauchy problem associated to the diffusion equation

$$\begin{cases} \partial_t u = d\Delta u & (t, X) \in (0; \infty) \times \mathbb{R}^N \\ u(0, X) = u_0(X) & X \in \mathbb{R}^N, \end{cases} \tag{I.2}$$

where $u_0 \in L^p\left(\mathbb{R}^N\right)$ with $1 \leq p \leq \infty$.

Solving Cauchy problem by Fourier transform

Assume as just said above that $N = 1$ and $p = 1$, we are going to use Fourier transform ([f]) to find a solution u (which we suppose existing) of (I.2). In the course of this development, we will make some assumptions on the solution – these assumptions can be verified later. Suppose first $u \in L^1(\mathbb{R})$ in order to use the Fourier transform. Let for $(t, x) \in [0; \infty) \times \mathbb{R}$,

$$\hat{u}(t, \xi) := \widehat{u(t, \bullet)}(\xi).$$

By assuming that $\partial_x u$ and $\partial_{xx} u$ are in $L^1(\mathbb{R})$ too, we can permute the Fourier transform and the derivative and thus the first line of (I.2) becomes

$$\partial_t \hat{u} = d\mathscr{F}[\partial_{xx} u].$$

Thanks to $\boxed{5}$ (page 117) it comes that $\hat{u}$ satisfies the following ODE-Cauchy problem where ξ states as a parameter:

$$\begin{cases} \hat{u}'(t, \xi) = -d\xi^2 \hat{u}(t, \xi) & t \in (0; \infty) \\ \hat{u}(0, \xi) = \hat{u_0}(\xi). \end{cases} \tag{I.3}$$

One solves the ODE of the latter problem, we get

$$\hat{u}(t, \xi) = \hat{u_0}(\xi) e^{-td\xi^2}.$$

We use then $\boxed{4}$ (page 117) with $a = 1/(4t)$:

$$\hat{u}(t, \xi) = \hat{u_0}(\xi) \cdot \underbrace{\mathscr{F}\left[\frac{1}{\sqrt{4\pi dt}} \exp\left(-\frac{\bullet^2}{4dt}\right)\right]}_{\substack{\text{Let us call} \\ \text{that } K(t,\bullet).}}(\xi).$$

One knows by $\boxed{2}$ (page 117) that the Fourier transform of a convolution product equals the pointwise product of the both Fourier transform, so

$$\hat{u}(t, \xi) = \mathscr{F}[K(t, \bullet) * u_0](\xi).$$

By finally exploiting the inversion formula $\boxed{3}$ (page 117), we obtain

$$u(t, x) = [K(t, \bullet) * u_0](x),$$

and it is now easy to check that our assumptions on u were correct.

[f] See the section "Fourier transform" (page 117) in the Toolbox part. The boxed numbers $\boxed{1}, \cdots, \boxed{5}$ refer to the five properties bearing on the Fourier transform properties.

Heat kernel on $\mathbb{R}^N$

The function K defined above by

$$K(t,x) = \frac{1}{\sqrt{4\pi dt}} \exp\left(-\frac{x^2}{4dt}\right)$$

is called the *heat kernel on* $\mathbb{R}$. This function owns a few remarkable properties:

- K verifies the heat equation with "initial datum" δ_0: the Dirac delta in 0.
- K is smooth in space for all time t positive, thereby the solution u inherits this property by convolution, even if u was not continuous at initial time. One calls this phenomenon the *regularizing effect of the heat equation.*
- K is positive for all time t positive. Again this property is also verified by the solution u thanks to the convolution, even if u was compactly supported at initial time. In the case where u_0 is compactly supported, one says that the support of *u spreads at infinite speed.*

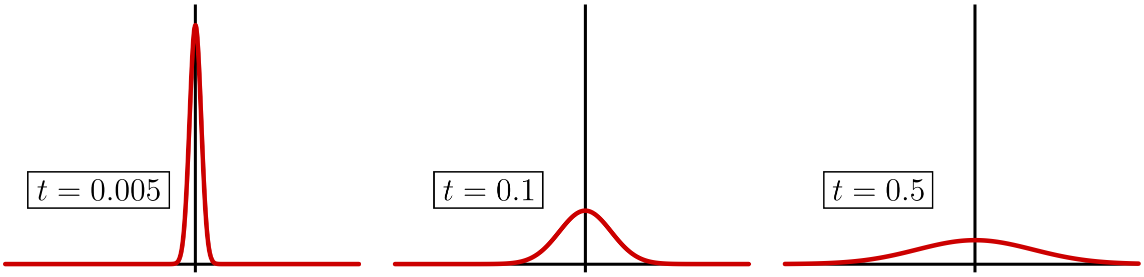


Figure F8 – *Snapshots of the the heat kernel on $\mathbb{R}$ at three different times.*

Definition 9 (Heat kernel on $\mathbb{R}^N$)
Let $N \in \mathbb{N}^\star$, one calls the *heat kernel on* $\mathbb{R}^N$ the function $K : \mathbb{R}_+^\star \times \mathbb{R}^N \to \mathbb{R}$ defined by the following

$$K(t,X) := \frac{1}{(4\pi dt)^{N/2}} \exp\left(-\frac{|X|^2}{4dt}\right).$$

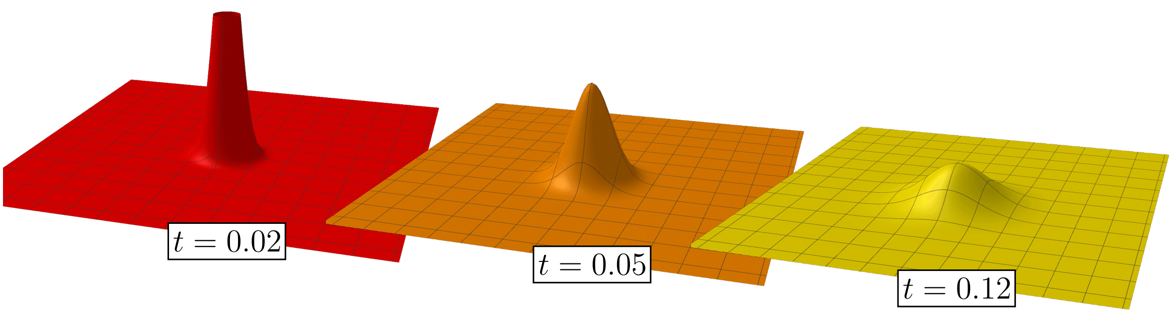


Figure F9 – *Snapshots of the the heat kernel on $\mathbb{R}^2$ at three different times.*

Remark. The heat kernel on $\mathbb{R}^N$ is an *approximate identity* in the following sense:

- For all positive time t, $K(t, \bullet)$ is positive.
- For all positive time t, $\|K(t, \bullet)\|_{L^1(\mathbb{R}^N)} = 1$.
- For all $\delta > 0$,

$$\lim_{t \to 0} \int_{|X| \geq \delta} K(t, X) \,\mathrm{d}X = 0.$$

The approximate identity is used for regularize by convolution.

Cauchy problem solution

One comes back to the Cauchy problem (I.2) of the diffusion equation in $\mathbb{R}^N$ $(N \in \mathbb{N}^*)$ provided with the initial datum $u_0 \in L^p\left(\mathbb{R}^N\right)$ $(1 \leq p \leq \infty)$. This problem is actually ill-posed because of the non-uniqueness of the solution. Indeed, take for example $N = 1$, $d = 1$ and $u_0 \equiv 0$, then the intuitive solution with such an initial datum should be $u \equiv 0$ (no population at initial time implies no population at any time). However, one can create another solution ([g]) which satisfies the same problem by posing $f(t) := e^{-1/t^2}$ and taking

$$u(t, x) := \begin{cases} \displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(t)\, x^{2n}}{(2n)!} & \text{if } t > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Despite this, we can get rid of the "bad" solutions whether we suppose *a priori* that the solution u does not increase too fast as $|X|$ tends to ∞. That is, by being more formal,

$$|u(t, X)| \leq A \exp\left(a |X|^2\right), \qquad \forall (t, X) \in (0; \infty) \times \mathbb{R}^N \qquad (\mathbf{I.4})$$

where A and a are real positive numbers. In the sequel, we shall work under this assumption in order to there only remains the one "physical solution". Thereby by "solution of (I.2)" we mean "solution of (I.2) which does not increase too fast".

Hypothesis 2: u does not increase too fast in the sense given by (I.4).

[g] Find out by Tychonoff (1935).

Theorem 10 (Solution of the diffusion Cauchy problem)
Let $N \in \mathbb{N}^\star$, $1 \leq p \leq \infty$, $u_0 \in L^p\left(\mathbb{R}^N\right)$. Then there exists a unique solution of the diffusion Cauchy problem (I.2) which is given by

$$u(t, X) = \begin{cases} [K(t, \bullet) * u_0](X) & \text{if } t > 0 \\ u_0(X) & \text{otherwise.} \end{cases}$$

This solution owns the following properties:

- $\boxed{\text{Smoothness}}$ $u \in \mathcal{C}^\infty\left((0;\infty) \times \mathbb{R}^N\right)$.
- $\boxed{\text{Integrability}}$ $u(t, \bullet) \in L^p\left(\mathbb{R}^N\right)$ for all $t \geq 0$ and
  - if $p \neq \infty$ then $u(t, \bullet) \xrightarrow[t \to 0]{L^p} u_0$,
  - if $p = \infty$ and u_0 is moreover continuous on $\mathbb{R}^N$, then
    - $u \in \mathcal{C}([0;\infty)) \times \mathbb{R}^N$; in other words, u is continuous in $t = 0$,
    - $u(t, X) \leq \|u_0\|_{L^\infty}$ for all $(t, X) \in [0;\infty) \times \mathbb{R}^N$,
  - if $p = 1$ then
    - $\|u(t, \bullet)\|_{L^1} = \|u_0\|_{L^1}$ for all $t \geq 0$,
    - $u(t, X) \leq (4\pi dt)^{-N/2} \|u_0\|_{L^1}$ for all $(t, X) \in [0;\infty) \times \mathbb{R}^N$.
- $\boxed{\text{Comparison}}$ Let u and v be the solutions of (I.2) with the respective initial datums u_0 and v_0,
  - if $u_0 \leq v_0$ on $\mathbb{R}^N$ then $u \leq v$ on $(0;\infty) \times \mathbb{R}^N$,
  - if $u_0 \leq v_0$ on $\mathbb{R}^N$ and $u_0 \not\equiv v_0$, then $u < v$ on $(0;\infty) \times \mathbb{R}^N$.

Remark. An important side of the diffusion equation is the mass conservation. The purpose of diffusion equation use is to model the movement of individuals in space; thus this tool should not change the population size – this role is played by the reaction presented in the previous section. One calls the *population mass* at time $t \geq 0$ the amount $\mathcal{M}(t) := \|u(t, \bullet)\|_{L^1(\mathbb{R}^N)}$. In population dynamics, it sounds acceptable to take a population with a finite positive initial mass, that is $\mathcal{M}(0) \stackrel{\text{def}}{=} \|u_0\|_{L^1(\mathbb{R}^N)} < \infty$, *i.e.* $u_0 \in L^1\left(\mathbb{R}^N\right)$. Then, thanks to the properties on u announced in the above theorem, we get $\mathcal{M}(t) = \mathcal{M}(0)$ for all $t \geq 0$, *i.e.* population mass does not change over time. Let us give a proof of this point for the one-dimensional case; we take some stronger assumptions than announced in the theorem by supposing the initial datum also non-negative and compactly supported – note that these additional guess are not embarrassing in the context of population study.

Proof (Mass conservation)
We work here in the space $\mathbb{R}$. Assume as previously advertised that $u_0 \in L^1(\mathbb{R})$ is a compactly supported non-negative function. We start by computing the two first derivatives in space of the heat kernel, there comes

$$\partial_x K(t,x) = C_1(d,t)\, x \exp\left(-\frac{x^2}{4dt}\right)$$

$$\partial_{xx} K(t,x) = C_2(d,t)\left(1-\frac{x^2}{2dt}\right)\exp\left(-\frac{x^2}{4dt}\right),$$

where C_1 and C_2 are two functions which do not depend on x. Let u be the solution of (I.2) with initial datum u_0 given by the theorem (10), one has

$$\partial_{xx} u(t,x) = \partial_{xx}\left([K(t,\bullet) * u_0](x)\right) = [\partial_{xx}(K(t,\bullet)) * u_0](x).$$

We check now whether $\partial_t u(t,\bullet)$ is in $L^1(\mathbb{R})$ which will allow us to permute the partial derivative in time and the integral. Let $t>0$,

$$\begin{aligned}\int_{\mathbb{R}} |\partial_t u(t,x)|\,\mathrm{d}x &= \int_{\mathbb{R}} |d\partial_{xx} u(t,x)|\,\mathrm{d}x \\ &\leq d\left|\int_{\mathbb{R}}\int_{\mathbb{R}} \partial_{xx}K(t,x-y)\,u_0(y)\,\mathrm{d}y\,\mathrm{d}x\right| \\ &= d\left|\int_{\mathbb{R}}\int_{\operatorname{supp}(u)} \partial_{xx}K(t,x-y)\,u_0(y)\,\mathrm{d}y\,\mathrm{d}x\right|\end{aligned}$$

then by Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}} |\partial_t u(t,x)|\,\mathrm{d}x &\leq d\left|\int_{\operatorname{supp}(u)} u_0(y)\int_{\mathbb{R}} \partial_{xx}K(t,x-y)\,\mathrm{d}x\,\mathrm{d}y\right| \\ &\leq d\max(u_0)\left|\int_{\operatorname{supp}(u)}\int_{\mathbb{R}} \partial_{xx}K(t,x-y)\,\mathrm{d}x\,\mathrm{d}y\right| \\ &= d\max(u_0)\left|\int_{\operatorname{supp}(u)} \|\partial_{xx}K(t,\bullet)\|_{L^1(\mathbb{R})}\,\mathrm{d}y\right| \\ &= d\max(u_0)\,\|\partial_{xx}K(t,\bullet)\|_{L^1(\mathbb{R})}\,|\operatorname{supp}(u)| \\ &< \infty.\end{aligned}$$

Therefore we have

$$\begin{aligned}\partial_t \|u(t,\bullet)\|_{L^1(\mathbb{R})} &= \partial_t \int_{\mathbb{R}} u(t,\bullet)\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \partial_t u(t,\bullet)\,\mathrm{d}x \\ &= d\int_{\mathbb{R}} \partial_{xx} u(t,\bullet)\,\mathrm{d}x \\ &= d\left[\partial_x u(t,+\infty) - \partial_x u(t,-\infty)\right] \\ &= 0.\end{aligned}$$

Thereby, $\|u(t,\bullet)\|_{L^1(\mathbb{R})}$ is constant with respect to the time and we get thus

$$\|u(t,\bullet)\|_{L^1(\mathbb{R})} = \|u_0\|_{L^1(\mathbb{R})}$$

for all $t \geq 0$, that is what we were aiming for. $\square$

I.3 Reaction-Diffusion in $\mathbb{R}^N$

We now combine the reaction equation (non-linear ODE) with the diffusion one (linear PDE) in $\mathbb{R}^N$ which brings us to the *reaction-diffusion equation* in $\mathbb{R}^N$ (non-linear PDE):

$$\partial_t u = d\Delta u + f(u).$$

This PDE models thus both the born/death of individuals in the population, and their scattering in the space $\mathbb{R}^N$. One couples it with some initial condition u_0 to get a Cauchy problem and one takes the following assumptions:

Hypothesis 3: $f(0) = 0$.

Hypothesis 4: $f \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$.

Hypothesis 5: u_0 is bounded and uniformly continuous in $\mathbb{R}^N$.

We shall not take any further hypothesis in this section in which we give some general results namely about existence, comparison principle and uniqueness.

I.3.1 Existence of a solution

We start by dealing with the existence of a solution for the reaction-diffusion Cauchy problem we mentioned above which we recall here:

$$\begin{cases} \partial_t u = d\Delta u + f(u) & (t,X) \in (0;\infty) \times \mathbb{R}^N \\ u(0,X) = u_0(X) & X \in \mathbb{R}^N. \end{cases} \tag{I.5}$$

Let us state a first result which guarantee the existence of a solution for (I.5) at least until some time T depending on u_0 and f.

Theorem 11 (Local well-posedness of the R-D Cauchy problem)
There exists some time $T = T\left(\|u_0\|_{L^\infty(\mathbb{R}^N)}, \mathrm{Lip}(f)\right) > 0$ such that (I.5) owns a solution $u : [0;T] \times \mathbb{R}^N \to \mathbb{R}$ satisfying the Duhamel's formula:

$$u(t,X) = [K(t,\blacksquare) * u_0](X) + \int_0^t [K(t-s,\blacksquare) * f(u(s,\blacksquare))](X)\,\mathrm{d}s.$$

Sketch of proof (Theorem 11)
Let $\tau > 0$, we pose

$$\mathcal{X} := \mathcal{C}^0\left([0;\tau], \mathrm{BUC}\left(\mathbb{R}^N\right)\right)$$

and we provide this space with the norm

$$\|u\|_{\mathcal{X}} := \sup_{0\leq t\leq\tau}\left\{\|u(t,\bullet)\|_{L^\infty(\mathbb{R}^N)}\right\}.$$

We are seeking (according to *Duhamel's principle*) a solution $(u : t \mapsto u(t,\bullet)) \in \mathcal{X}$ for (I.5) of this shape:

$$u(t,\bullet) = [K(t,\blacksquare) * u_0](\bullet) + \int_0^t [K(t-s,\blacksquare) * f(u(s,\blacksquare))](\bullet)\,\mathrm{d}s.$$

Note that the unknown u is present in the both hands of the latter equality; the problem lies therefore in finding a fixed point. First of all, remark that

- $[0;\tau]$ is compact, and
- $\left(\mathrm{BUC}\left(\mathbb{R}^N\right), \|\bullet\|_{L^\infty(\mathbb{R}^N)}\right)$ is a Banach space

imply that $(\mathcal{X}, \|\bullet\|_{\mathcal{X}})$ is also a Banach space. Then one takes

$$\Gamma := \left\{u \in \mathcal{X} \;/\; \forall t \in [0;\tau), \|u(t,\bullet) - [K(t,\blacksquare) * u_0(\blacksquare)](\bullet)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}\right\} \subsetneq \mathcal{X},$$

that is Γ is a set of functions which stay "not too far" in the $\|\bullet\|_{L^\infty}$ sense from the homogeneous solution of (I.5) – *i.e.* the solution without reaction. One verifies that Γ is closed in the Banach space $(\mathcal{X}, \|\bullet\|_{\mathcal{X}})$ and so Γ is complete for $\|\bullet\|_{\mathcal{X}}$. We pose then

$$\Phi : \left\{\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X} \\ u & \longmapsto & \Phi u : \left\{\begin{array}{ccc} [0;\tau] & \longrightarrow & \mathrm{BUC}\left(\mathbb{R}^N\right) \\ t & \longmapsto & \Phi u(t,\bullet), \end{array}\right. \end{array}\right.$$

where

$$\Phi u(t,\bullet) := [K(t,\blacksquare) * u_0](\bullet) + \int_0^t [K(t-s,\blacksquare) * f(u(s,\blacksquare))](\bullet)\,\mathrm{d}s,$$

and we proceed by using a *fixed point theorem* on the map $\Phi_{|\Gamma}$.

Theorem 12 (Banach-Picard fixed point)
Let Γ be a non-empty complete metric space and Φ a contraction on Γ, then Φ admits a unique fixed point.

The end of the proof is therefore to check that the Banach-Picard fixed point theorem assumptions are satisfied, then one has to show that:

- $\Phi_{|\Gamma} : \Gamma \to \Gamma$, otherwise said, $\operatorname{Im}\left(\Phi_{|\Gamma}\right) = \Gamma$. It comes out of the demonstration of this point that τ has to be smaller or equal to

$$T := \frac{1}{2 \operatorname*{Lip}_I (f)} \qquad \textbf{(I.6)}$$

where $I = [-2\|u_0\|_{L^\infty} ; 2\|u_0\|_{L^\infty}]$.

- $\Phi_{|\Gamma}$ is a contraction on Γ.

Observe that the Banach-Picard fixed point theorem (12) give us existence and uniqueness of a solution for (I.5) but only in Γ which is strictly include in $\mathcal{X}$; therefore it might happen that some other solutions exists in $\mathcal{X}$, that's why the local well-posedness theorem 11 only gives the solution existence. Actually, we may pursue to show uniqueness, but we shall later give another proof which will be an immediate corollary of comparison principle. □

Theorem 13 (Global well-posedness of the R-D Cauchy problem)
Suppose we know *a priori* there is some $M > 0$ such that $|u|$ cannot exceed M whenever it exists. Then the local solution of (I.5) guaranteed by the local well-posedness theorem (11) is actually global, *i.e.* $u : [0; \infty) \times \mathbb{R}^N \to \mathbb{R}$.

Remark. The latter theorem give some sufficient condition to obtain a global solution. Another one is to have f globally Lipschitz.

Sketch of proof (Theorem 13)
The main idea of the proof is to repeat the local well-posedness theorem (11) by taking as initial datum for each new iteration, the solution at last time of the previous one. Then by plugging the solutions successively we get the global solution we are aiming for.

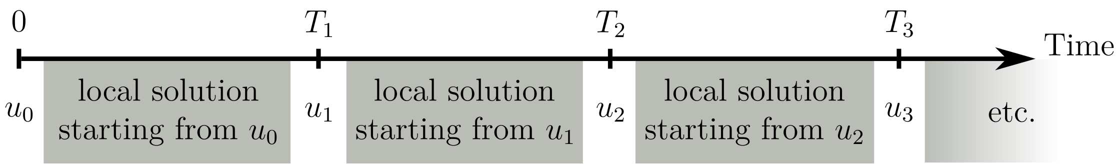


Figure F10 – *Illustration of the construction of a global well-posedness.*

The risk in doing this is that the lifetime of the local solutions may decrease faster than the general term of some convergent series. Hence, we could not reach an infinite time.

Figure F11 – *Illustration of one case which does not allow us to claim the global solution existence.*

Here comes the hypothesis of boundedness of $|u|$. Let $\tau \geq 0$ be a time whom we know u exists at it. We have,

$$I_\tau := \left[-2\left\|u(\tau, \bullet)\right\|_{L^\infty}; 2\left\|u(\tau, \bullet)\right\|_{L^\infty}\right] \subset [-2M; 2M] := J.$$

Whence $\underset{I_\tau}{\mathrm{Lip}}(f) \leq \underset{J}{\mathrm{Lip}}(f)$ and thus, by calling T_τ the lifetime of the local solution starting at time τ and using (I.6) page 20,

$$T_\tau \overset{\text{def}}{=} \frac{1}{2\underset{I_\tau}{\mathrm{Lip}}(f)} \geq \frac{1}{2\underset{J}{\mathrm{Lip}}(f)}.$$

The rightmost hand of the latter is positive and independent of τ. Therefore, the risk mentioned above can be excluded. □

I.3.2 Comparison principle

Consider now the *parabolic differential operator*

$$\mathcal{L}u := \partial_t u - d\Delta u - f(u)$$

with which the PDE $\partial_t u = d\Delta u + f(u)$ becomes $\mathcal{L}u = 0$.

Definition 14 (Sub-solution)
Let $\underline{u} \in \mathcal{C}^{1,2}\left([0;T) \times \mathbb{R}^N, \mathbb{R}\right)$, one says that $\underline{u}$ is a *sub-solution* for the operator $\mathcal{L}$ if

$$\mathcal{L}\underline{u} \leq 0$$

for all $(t, X) \in [0;T) \times \mathbb{R}^N$.

Definition 15 (Super-solution)
Let $\overline{u} \in \mathcal{C}^{1,2}\left([0;T) \times \mathbb{R}^N, \mathbb{R}\right)$, one says that $\overline{u}$ is a *super-solution* for the operator $\mathcal{L}$ if

$$\mathcal{L}\overline{u} \geq 0$$

for all $(t, X) \in [0;T) \times \mathbb{R}^N$.

Remark. Note that $u \in \mathcal{C}^{1,2}\left([0;T) \times \mathbb{R}^N, \mathbb{R}\right)$ is a solution for the operator $\mathcal{L}$ if and only if u is both sub- and super-solution.

Theorem 16 (Parabolic non-linear comparison principle)
Let $\underline{u}, \overline{u} \in \mathcal{C}^{1,2}\left([0;T) \times \mathbb{R}^N, \mathbb{R}\right)$ and u be a solution of the Cauchy problem (I.5) which we recall here using the $\mathcal{L}$ notation,

$$\begin{cases} \mathcal{L}u = 0 & (t, X) \in (0;\infty) \times \mathbb{R}^N \\ u(0, X) = u_0(X) & X \in \mathbb{R}^N. \end{cases}$$

$\boxed{\text{Lower comparison}}$

$$\left[\begin{array}{c} \underline{u} \text{ is a sub-solution} \\ \underline{u}(0, \bullet) \leq u_0 \end{array}\right] \Rightarrow [\, \underline{u} \leq u \,].$$

$\boxed{\text{Upper comparison}}$

$$\left[\begin{array}{c} \overline{u} \text{ is a super-solution} \\ \overline{u}(0, \bullet) \geq u_0 \end{array}\right] \Rightarrow [\, \overline{u} \geq u \,].$$

$\boxed{\text{Strict lower comparison}}$

$$\left[\begin{array}{c} \underline{u} \text{ is a sub-solution} \\ \underline{u}(0, \bullet) \leq u_0 \\ \underline{u}(0, \bullet) \not\equiv u_0 \end{array}\right] \Rightarrow [\, \underline{u} < u \,].$$

$\boxed{\text{Strict upper comparison}}$

$$\left[\begin{array}{c} \overline{u} \text{ is a super-solution} \\ \overline{u}(0, \bullet) \geq u_0 \\ \overline{u}(0, \bullet) \not\equiv u_0 \end{array}\right] \Rightarrow [\, \overline{u} > u \,].$$

Corollary 17 (of theorem 16) (Uniqueness of the solution)
The local solution of the reaction-diffusion Cauchy problem (I.5) given by theorem 11 is unique.

Proof (Corollary 17)
Let u and $\tilde{u}$ be two solutions of (I.5) starting from the same initial condition u_0. It is clear that

- $\mathcal{L}\tilde{u} = 0 \leq 0$
- $\tilde{u}(0, \bullet) = u_0 \leq u_0$,

hence by comparison, $\tilde{u} \leq u$. Then repeat the latter by inserting "$\geq$" instead of "$\leq$" to get $\tilde{u} \geq u$ and thereby $\tilde{u} \equiv u$. $\square$

Corollary 18 (of theorem 16) (About reaction equilibriums)
Let $u_E \in \mathbb{R}$ be an equilibrium point for the reaction, *i.e.* $f(u_E) = 0$ and u be the solution of (I.5) starting from u_0.

- If $u_0 \equiv u_E$, then $u \equiv u_E$.
- If $u_0 \leq u_E$, then $u \leq u_E$.
- If $u_0 \geq u_E$, then $u \geq u_E$.
- If $u_0 \leq u_E$ but $\not\equiv u_E$, then $u < u_E$.
- If $u_0 \geq u_E$ but $\not\equiv u_E$, then $u > u_E$.

Proof (Corollary 17)
One just needs to see that

$$\mathcal{L}u_E = \underbrace{\partial_t u_E - d\Delta u_E}_{\substack{=0 \text{ because } u_E \in \mathbb{R} \text{ is} \\ \text{independant of } t \text{ and } X}} - \underbrace{f(u_E)}_{\substack{=0 \text{ because } u_E \text{ is} \\ \text{an equilibirum point}}} = 0.$$

The results are just then some consequences from the comparison principle. □

Remark. By taking the carrying capacity K (see page 8) equals 1, each of the three following models

- logistic,
- monostable degenerate,
- bistable

owns at least two equilibrium points at $\underline{u_E} = 0$ and $\overline{u_E} = 1$. Therefore, by taking the initial datum u_0 between these two points, we are *a priori* sure that the local solution of the R-D Cauchy problem (I.5) stay bounded by between 0 and 1. One can then apply the global well-posedness theorem 13: the local solution is actually global.

Corollary 19 (of theorem 16) (If u_0 is sub-solution of its own PDE)
Always considering the R-D Cauchy problem (I.5), assume furthermore that u_0 is a sub-solution for $\mathcal{L}$. Then for all given $X \in \mathbb{R}^N$, $u(\bullet, X)$ is an *increasing* function.

Proof (Corollary 17)
Let X be fixed in $\mathbb{R}^N$ and $t, \tau \geq 0$, we pose

$$w(t, X) := u(t + \tau, X).$$

The goal of this proof is showing that $w \geq u$. We do this thanks to the comparison principle:

- from one part, it is clear that w is a solution (and thus a super-solution) of $\mathcal{L}$,
- from another part, since u_0 is a sub-solution for $\mathcal{L}$ and $u_0 \leq u_0$, we get by comparison principle,

  $$u_0(X) \leq u(t, X) \text{ for all } t \geq 0.$$

  Rewriting the latter with $t = 0 + \tau$, it comes

  $$u_0(X) \leq u(0 + \tau, X) \stackrel{\text{def}}{=} w(0, X) =: w_0(X).$$

Hence by applying the comparison principle a second time, one finally obtains

$$w(t, X) \stackrel{\text{def}}{=} u(t + \tau, X) \geq u(t, X) \qquad \forall t, \tau \geq 0,$$

that is $u(\bullet, X)$ is an increasing function for all $X \in \mathbb{R}^N$. □

Remark. Corollary 19 owns also an antagonistic version: if u_0 is a super-solution for $\mathcal{L}$, then for all given $X \in \mathbb{R}^N$, $u(\bullet, X)$ is an *decreasing* function. The proofs of the both versions are exactly the same.

I.4 Fisher-KPP equation

We work in this section on the equations of the type *Fisher-KPP* ([h]) in $\mathbb{R}^N$ which are a particular case of reaction-diffusion equations whom the reaction function f satisfies the following properties (and is thus said "of KPP type"):

- $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$,
- f is non-negative,
- $f(0) = f(1) = 0$ so that 0 and 1 are some equilibrium points for the ODE $u' = f(u)$,

[h] Both Fisher and "KPP" (abbreviation for "Kolmogorov, Petrovsky, Piskunov") worked independently on these equations.

- $f'(1) < 0 < f'(0)$ so that 0 is not asymptotically stable and 1 is asymptotically stable,
- f satisfies the KPP-hypothesis (see page 8), that is

$$f(u) \leq uf'(0) \qquad \forall u \geq 0.$$

Remark. The logistic reaction (see page 7) taken with its carrying capacity $K = 1$, $f(u) = ru(1-u)$, is of KPP type.

Ones recalls that by taking an initial datum u_0 between the both equilibriums 0 and 1, the R-D Cauchy problem

$$\begin{cases} \partial_t u = d\Delta u + f(u) & (t, X) \in (0; \infty) \times \mathbb{R}^N \\ u(0, X) = u_0(X) & X \in \mathbb{R}^N \end{cases} \tag{I.7}$$

owns a unique global solution $u : [0; \infty) \times \mathbb{R}^N$. Now we know the existence of such a solution a legitimate question could be: "what happen when t becomes large ?" Indeed, there is a competitive relation between the reaction term and the diffusion one.

- The reaction term $f(u)$ tends to raise the population size, especially when the latter is small; the PDE without diffusion

$$\partial_t u = f(u)$$

  leads to a space invasion to the steady state $u_\infty \equiv 1$ as soon as $u_0 > 0$.

- The diffusion term $d\Delta u$ spreads the individuals in space; the PDE without reaction

$$\partial_t u = d\Delta u$$

  leads to the extinction of the population, that is u tends to the steady state $u_\infty \equiv 0$.

In the Fisher-KPP equations case, the reaction prevails: *all positive perturbation of the steady state $u_0 \equiv 0$ leads to a space invasion toward $u_\infty \equiv 1$.* More precisely, we have

Theorem 20 (Hair Trigger Effect)
For all initial datum $0 \leq u_0 \leq 1$ but $\not\equiv 0$ and all $R > 0$, the solution u of the Fisher-KPP Cauchy problem (I.7) satisfies

$$\lim_{t\to\infty} \left(\inf_{|X|\leq R} \{u(t, X)\} \right) = 1,$$

and thus, $u(t, \bullet)$ tends to the constant function $u_\infty \equiv 1$ locally uniformly in space as t tends to $+\infty$.

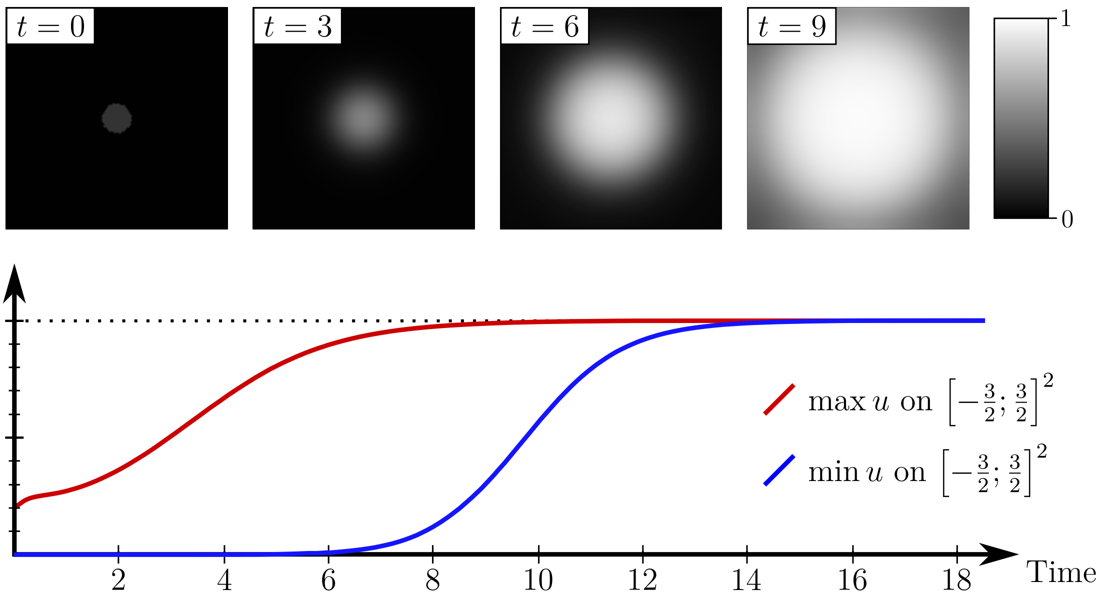


Figure F12 – *Simulation of the Hair Trigger Effect in* $\mathbb{R}^2$ *using the logistic reaction* $f(u) = u(1-u)$ *and starting from the compactly supported initial datum* $u_0 := \frac{\mathbb{1}_{\mathcal{B}_{1/5}}}{5}$. *On the top of the figure, one sees four snapshots of the solution through the window* $\left[-\frac{3}{2};\frac{3}{2}\right]^2$, *and on the bottom we have represented the local* min *and* max *of this solution in the same window.*

Remark. About the proof of the theorem 20, it is sufficient to check that the result is true for the logistic reaction $ru(1-u)$. Indeed, let f be any reaction function of KPP type, by taking $r > 0$ close enough to 0, we may have $ru(1-u) < f(u)$ as one sees in the figure (*F13*) above. Then the solution $\underline{u}$ of problem (I.7) using $ru(1-u)$ instead of $f(u)$ is a sub-solution for $\partial_t u = d\Delta u + f(u)$ and satisfies $\underline{u}_0 \leq u_0$; so by comparison principle, we get $\underline{u} \leq u$ and thanks to the hair trigger effect on $\underline{u}$, u is driven to the steady state $u_\infty \equiv 1$ locally uniformly in space too.

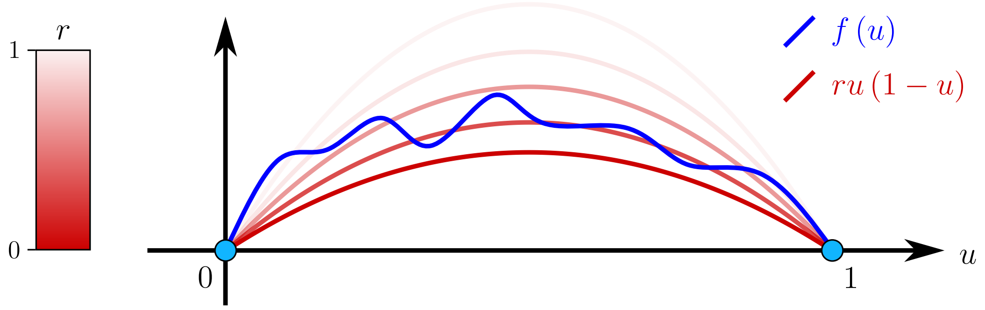


Figure F13 – *Illustration of what one means by "$r > 0$ close enough to 0" so that* $ru(1-u)$ *be bellow* $f(u)$ *in the previous remark.*

Proof (Theorem 20) (HTE)

We therefore show the result for the logistic reaction function $f(u) = ru(1-u)$ with $r > 0$. Let us define the differential operator $\mathcal{L}$ by this way:

$$\mathcal{L}u := \partial_t u - d\Delta u - ru(1-u).$$

For $R > 0$, we denote by (λ_R, φ_R) the couple principal-(eigenvalue/eigenfunction) of $-d\Delta$ with Dirichlet boundary condition in $\mathcal{B}_R$ as defined in section 2 of the Toolbox part page 118. We proceed like that:

$\boxed{\mathbf{1}}$ One shows that for R large enough and $\varepsilon > 0$ small enough, the function

$$\underline{u}_0(X) := \begin{cases} \varepsilon\varphi_R(X) & \text{if } X \in \mathcal{B}_R \\ 0 & \text{otherwise} \end{cases}$$

is a sub-solution for $\mathcal{L}$.

We set $\underline{u}$ the solution of the PDE $\mathcal{L}u = 0$ starting from the initial datum $\underline{u}_0$.

$\boxed{\mathbf{2}}$ Because u_0 may be compactly supported, we wait for the u peeling off due to the diffusion in order to (up to choosing ε smaller) slip $\underline{u}$ under $u(t=1,\bullet)$.

Thanks to $\boxed{\mathbf{2}}$, we deduce by comparison principle that $u \geq \underline{u}$. The fact that $\underline{u}_0$ is sub-solution of its own equation implies (see corollary 4 page 24) that $\underline{u}_0$ is increasing in time for $X \in \mathbb{R}^N$ fixed. Therefore, since u is bounded by 1, it converges pointwise to a stationary function p depending only on X and such that $0 < p(X) \leq 1$. Then parabolic estimates allows us to say that the convergence pointwise of u toward p is actually locally uniform; thus we can "take the limit in the equation" so that p satisfies the following elliptic PDE

$$-d\Delta p = rp(1-p). \qquad \textbf{(I.8)}$$

$\boxed{\mathbf{3}}$ It remains to show that the only positive solution of (I.8) smaller than 1 is $p \equiv 1$.

$\boxed{\mathbf{1}}$ Let's show that $\underline{u}_0$ is a sub-solution for $\mathcal{L}$. We set $\varepsilon \in (0;1)$, for X inside $\mathcal{B}_R$, we have

$$\begin{aligned} \mathcal{L}(\varepsilon\varphi_R) &= -d\varepsilon\Delta\varphi_R - r\varepsilon\varphi_R(1-\varepsilon\varphi_R) \\ &= \varepsilon\varphi_R(d\lambda_R - r(1-\varepsilon\varphi_R)) \end{aligned}$$

one had choosen φ_R such that $\|\varphi_R\|_{L^\infty} = 1$, so

$$\mathcal{L}(\varepsilon\varphi_R) \leq \varepsilon(d\lambda_R - r(1-\varepsilon))$$

since λ_R tends to 0 as R tends to $+\infty$, we can choose R large enough so that λ_R is smaller than $r(1-\varepsilon)/2d$; then there comes

$$\mathcal{L}(\varepsilon\varphi_R) \leq -\frac{r\varepsilon(1-\varepsilon)}{2} \leq 0.$$

For X outside $\mathcal{B}_R$, one easy sees that $\mathcal{L}(0) = 0$, thereby $\underline{u}_0$ is a sub-solution for $\mathcal{L}$.

$\boxed{\mathbf{2}}$ One knows that u_0 is non-negative and non-zero. Since 0 is a sub solution we get, thanks to strict comparison principle,

$$u(t,X) > 0, \qquad \forall (t,X) \in (0;\infty) \times \mathbb{R}^N.$$

Then by taking $t = 1$ and choosing $\varepsilon := \min\left(1, \inf_{\mathbb{R}^N}\{u(1,\bullet)\}\right)$ we get $\underline{u} \leq u(1+t,\bullet)$. Then, as said in the beginning of the proof, u converges locally uniformly toward a function p which satisfies

- $0 \leq p \leq 1$,
- $p \not\equiv 0$
- p satisfies the the elliptic PDE (I.8).

$\boxed{\mathbf{3}}$ Let p be a positive solution of (I.8) smaller than 1 and suppose by contradiction that $p \not\equiv 1$. Then there exists some point X_0 in $\mathbb{R}^N$ where $0 < p(X_0) < 1$. Without loss generalities, one can assume that X_0 is 0 (otherwise, take the translation $\widetilde{p}(X) := p(X+X_0)$ which satisfies also (I.8)). Under this absurd hypothesis, one starts by showing that $\varepsilon\varphi_R$ is a *strict* sub-solution of

$$\widetilde{\mathcal{L}}p := -d\Delta p - rp(1-p)$$

for R large enough and ε small enough:

$$\begin{aligned} \widetilde{\mathcal{L}}(\varepsilon\varphi_R) &= \varepsilon\varphi_R(d\lambda_R - r(1-\varepsilon\varphi_R)) \\ &\leq \varepsilon(d\lambda_R - r(1-\varepsilon)) \end{aligned}$$

then by choosing $\varepsilon \leq p(0)$,

$$\widetilde{\mathcal{L}}(\varepsilon\varphi_R) \leq p(0)(d\lambda_R - r(1-p(0)))$$

and finally take R large enough to have $\lambda_R \leq r(1-p(0))/2d$ (note that R does not depend on ε),

$$\widetilde{\mathcal{L}}(\varepsilon\varphi_R) \leq -\frac{rp(0)\overbrace{(1-p(0))}^{\substack{>0 \text{ thanks} \\ \text{to the hyp}}}}{2} < 0. \qquad \textbf{(I.9)}$$

Now, because p is positive, the set

$$\mathcal{E} := \{\varepsilon > 0 \ / \ \forall X \in \mathcal{B}_R, \quad \varepsilon\varphi_R(X) \leq p(X)\}$$

is non-empty and bounded by $p(0)$ (since $\varepsilon\varphi_R(0) = \varepsilon$); so the amount $\varepsilon^* := \sup \mathcal{E}$ exists. Let us call $X^* \in \mathcal{B}_R$ the contact point between $\varepsilon^*\varphi_R$ and p. The figure (*F14*) bellow summarizes the situation.

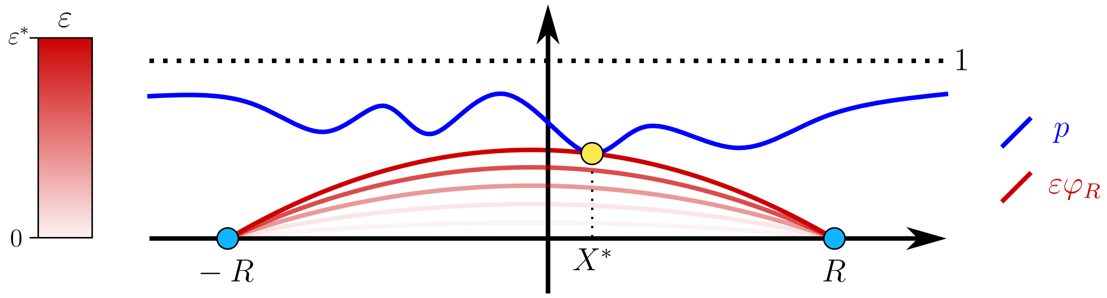


Figure F14 – *One increases ε until $\varepsilon\varphi_R$ "touches" the function p.*

We have, from one part, because $\varepsilon^*\varphi_R - p$ reaches a maximum in $X^* \in \mathcal{B}_R$,

$$\Delta(\varepsilon^*\varphi_R - p)(X^*) \leq 0,$$

from another part, by using (I.8), (I.9) that $\varepsilon^*\varphi_R = p$ in X^*, we get

$$\Delta(\varepsilon^*\varphi_R - p)(X^*) > 0$$

which is absurd, and thus $p \equiv 1$. $\square$

I.5 Weak Allee effect

We consider now a monostable degenerate reaction in the R-D Cauchy problem which becomes (the carrying capacity K is taken equals 1)

$$\begin{cases} \partial_t u = d\Delta u + ru^{1+p}(1-u) & (t,X) \in (0;\infty) \times \mathbb{R}^N \\ u(0,X) = u_0(X) & X \in \mathbb{R}^N \end{cases} \qquad \textbf{(I.10)}$$

where p is a real positive number and u_0 is such that

- $0 \leq u_0 \leq 1$,
- $u_0 \not\equiv 0$,
- $u_0 \not\equiv 1$.

Firstly, we shall use a quite different reaction by removing the term $(1-u)$ and thus work on the problem:

$$\begin{cases} \partial_t u = d\Delta u + ru^{1+p} & (t,X) \in (0;\infty) \times \mathbb{R}^N \\ u(0,X) = u_0(X) & X \in \mathbb{R}^N. \end{cases} \tag{I.11}$$

Note that $u_\infty = 1$ is no longer an equilibrium state of the reaction, thereby it could happen that the local solution of (I.11) given by theorem 11 is not global and blows up in a finite time.

I.5.1 Fujita's blow up

The question one sees here is knowing whether the solution u of (I.11) is global or blows up in a finite time. There is actually a competition between the both terms $d\Delta u$ and ru^{1+p}, in a relative same way as the remark we have done page 25:

- The reaction term ru^{1+p} pull the solution to the blow-up in a finite time.
- The diffusion term $d\Delta u$ tend to crush u on zero by spreading the individuals in space.

An obvious case is the one where u_0 is everywhere larger than some $\varepsilon > 0$. Indeed, under this assumption we can easily make a sub-solution which push u to the blow-up: take $\underline{u} = \underline{u}(t)$ the solution of the ODE-Cauchy problem

$$\begin{cases} \underline{u}'(t) = \underline{u}^{1+p}(t) & t \in (0;\infty) \\ \underline{u}(0) = \varepsilon. \end{cases}$$

One can show from one part that

$$\underline{u}(t) = \left(\frac{1}{\varepsilon^p} - pt\right)^{-1/p}$$

then blows up in a finite time, and from another part, $\underline{u}$ is below u thanks to the comparison principle. Therefore, u blows up in a finite time too.

Now we have dealt with this case, we are interested in the sequence in compactly supported initial datums.

Theorem 21 (Fujita, 1966)
Let define the Fujita's exponent $p_F := 2/N$. ([a])

Soft Allee effect If $0 < p \leq p_F$, then for all non-negative and non-zero initial datum the solution of (I.11) blows up in a finite time.

Hard Allee effect If $p > p_F$, then there exists some "small enough" non-negative initial datums which give global solutions for (I.11) tending then to 0 as t tends to $+\infty$.

[a] Recall that N is the space dimension.

Remark. Otherwise said, the Fujita's theorem provides a threshold intensity of Allee effect p_F which splits *systematic blow up* due to a soft Allee effect ($p \leq p_F$) from *non-systematic blow up* due to an intense Allee effect ($p > p_F$).

Lemma 22 (On the decreasing speed of the heat solutions)
Let u_0 be a non-negative and non-zero initial datum and denote u the solution of the heat equation $\partial_t u = d\Delta u$ in $\mathbb{R}^N$. One controls the L^∞ norm of $u(t, \bullet)$ thanks to the following equality:

$$\|u(t, \bullet)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{(1+t)^{N/2}}.$$

Where $C = C(\|u_0\|_{L^\infty}, \|u_0\|_{L^1}) := \max\left(\left(\frac{1}{2\pi d}\right)^{N/2} \|u_0\|_{L^1}, 2^{N/2}\|u_0\|_{L^\infty}\right)$.

Proof (Lemma 22)
First of all, it is easy to see that

$$\|u(t, \bullet)\|_{L^\infty} \leq \min\left(\|u_0\|_{L^\infty}, \frac{\|u_0\|_{L^1}}{(4\pi dt)^{N/2}}\right). \qquad (\bigstar)$$

We split in two cases:

$\boxed{\text{If } t \geq 1}$ then let us pose the amount $k := 1/(2\pi d)$; there comes,

$$4\pi dkt = 2t \geq 1 + t$$

whence

$$\frac{k^{N/2}\|u_0\|_{L^1}}{(1+t)^{N/2}} \geq \frac{k^{N/2}\|u_0\|_{L^1}}{(4\pi dkt)^{N/2}} = \frac{\|u_0\|_{L^1}}{(4\pi dt)^{N/2}} \overset{(\bigstar)}{\geq} \|u(t, \bullet)\|_{L^\infty}.$$

$\boxed{\text{If } t < 1}$ then $1 + t < 2$ and so

$$\frac{2^{N/2}}{(1+t)^{N/2}} > 1,$$

whence

$$\frac{2^{N/2}\|u_0\|_{L^\infty}}{(1+t)^{N/2}} > \|u_0\|_{L^\infty} \overset{(\bigstar)}{\geq} \|u(t, \bullet)\|_{L^\infty}.$$

Thereby, by posing C equals to the max of the both numerators of each case, we finally get the expected result:

$$\|u(t, \bullet)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{(1+t)^{N/2}}. \quad \square$$

Proof (Theorem 21) (Fujita) $\boxed{\textbf{Hard Allee effect}}$

Let p be such that $p > p_F$, we aim to build a global super-solution for

$$\mathcal{L}u := \partial_t u - d\Delta u - u^{1+p}$$

which tends to $u_\infty \equiv 0$ as t tends to $+\infty$. One tries for this a function of this shape:

$$\overline{u}(t, X) := v(t, X) \cdot g(t),$$

where v is the solution of the heat equation $\partial_t v = d\Delta v$ starting from u_0 and $g : [0; \infty) \to \mathbb{R}_+^*$ is to be determined. We start by impose $g(0) = 1$ so that $\overline{u}(0, \bullet) = u_0$ and therefore the initial datum question is settled. We demand now that $\overline{u}$ is a super-solution for the operator $\mathcal{L}$. One has

$$\begin{aligned}\mathcal{L}\overline{u} &= \partial_t \overline{u} - d\Delta \overline{u} - \overline{u}^{1+p} \\ &= g\partial_t v + vg' - gd\Delta v - g^{1+p}v^{1+p} \\ &= g(\partial_t v - d\Delta v) + vg' - g^{1+p}v^{1+p}\end{aligned}$$

using then that v is solution of the heat equation,

$$\mathcal{L}\overline{u} = vg' - g^{1+p}v^{1+p},$$

we thus get

$$\mathcal{L}\overline{u}(t, X) = \underbrace{v(t, X)}_{\geq 0} \underbrace{\Big(g'(t) - g^{1+p}(t)\, v^p(t, X)\Big)}_{\text{We would like this amount non-negative too...}}.$$

Recalling the L^∞ control on v given in lemma 22, we look for a function g satisfying the ordinary differential inequation

$$g'(t) \geq g^{1+p}(t) \frac{C^p}{(1+t)^{Np/2}}.$$

Solving the ODE Cauchy problem associated to the later starting from 1 at initial time, one finds,

$$\frac{1}{g^p(t)} = \begin{cases} 1 + \dfrac{p\, C^p}{\frac{Np}{2} - 1}\left(\dfrac{1}{(1+t)^{\frac{Np}{2}-1}} - 1\right) & \text{if } p \neq \frac{2}{N}, \\ \\ 1 - \dfrac{2}{N} C^{\frac{2}{N}} \ln(1+t) & \text{if } p = \frac{2}{N} \end{cases}$$

which should not be zero otherwise g would blow up in a finite time.

$\boxed{\text{If } p \leq 2/N}$ then $1/g^p$ tends to $-\infty$ as t tends to $+\infty$. Therefore, because $1/g^p$ starts from 1, it hits 0 at some positive time, that is g blows up in finite time and we cannot make a function g in this way.

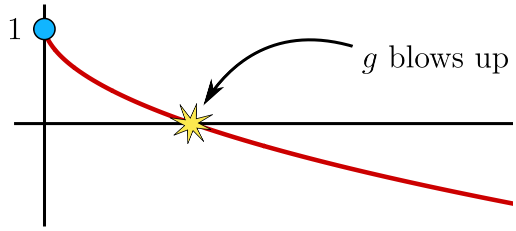


Figure F15 – *Graph of $1/g^p$ given with $N = 2$ and $p = 0.5$. The function g systematically blows up in finite time.*

$\boxed{\text{If } p > 2/N}$ then

$$\lim_{t\to+\infty} \frac{1}{g^p(t)} = 1 - \frac{pC^p}{\frac{Np}{2} - 1}$$

which might be positive whether $C = C\left(\|u_0\|_{L^\infty}, \|u_0\|_{L^1}\right)$ were chosen "small enough". Therefore such a function g may be suitable for our super-solution $\overline{u} = vg$.

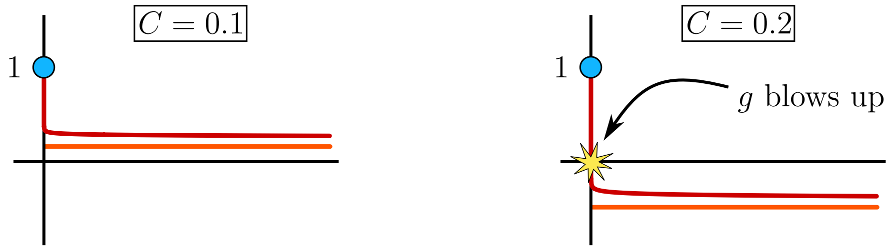


Figure F16 – *Graph of $1/g^p$ given with $N = 2$ and $p = 1.1$. The value $C = 0.1$ is "small enough" so that g does not blows up in finite time.*

Consequently, if $p > 2/N$ and u_0 is taken such that $\|u_0\|_{L^\infty}$ and $\|u_0\|_{L^1}$ are "small enough", we can take the function $\overline{u}$ as a super-solution for $\mathcal{L}$. We thus get, thanks to the comparison principle,

$$0 \leq u(t, X) \leq \overline{u}(t, X) \overset{\text{def}}{=} \underbrace{v(t, X)}_{\substack{\text{decays to } 0 \\ \text{like } ct^{-N/2}}} \cdot \overbrace{g(t)}^{\text{bounded}} \xrightarrow{t\to+\infty} 0,$$

that is the solution u is global and tends to the stationary state $u_\infty \equiv 0$ at speed $ct^{-N/2}$. $\square$

Proof (Theorem 21) (Fujita) $\boxed{\textbf{Soft Allee effect}}$

We will only treat here the non-degenerate cases $0 < p < p_F \overset{\text{def}}{=} 2/N$. Let u_0 be a non-negative and non-zero initial datum, we are willing to prove that the solution u starting from u_0 blows up in finite time. To do this, assume by contradiction that u is a global solution of (I.11). We take the following function

$$f(t) := \int_{\mathbb{R}^N} K(t,X)\, u_0(X)\, \mathrm{d}X,$$

where K denotes the heat kernel in $\mathbb{R}^N$ we have defined second section (see page 14). We are going to surround f by two functions then we will show that the order relation between the both bounds is conflicting once t becomes large.

$\boxed{\text{Lower bound}}$ Let δ and ε be some real positive numbers such that $\delta \mathbb{1}_{\mathcal{B}(X_0,\varepsilon)} < u_0$, where X_0 is in $\mathrm{supp}\,(u_0)$, like it is summarize on figure (*F17*).

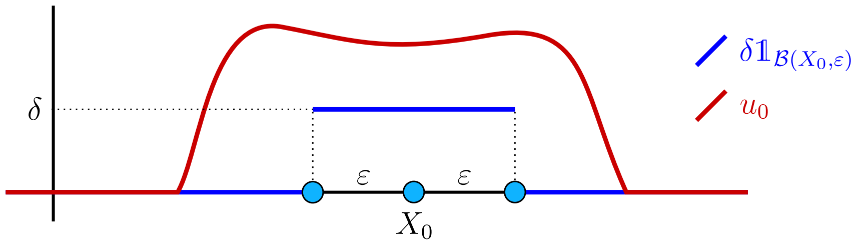


Figure F17 – *Illustration of the situation.*

We multiply by K the previous inequality then integrate on $\mathbb{R}^N$ to get the lower bound:

$$f(t) \geq \delta \int_{\mathcal{B}(X_0,\varepsilon)} K(t,X)\, \mathrm{d}X = \frac{C}{t^{N/2}}. \qquad \textbf{(I.12)}$$

$\boxed{\text{Upper bound}}$ For $0 \leq s \leq t$, one poses

$$g(s) := \int_{\mathbb{R}^N} K(t-s,X)\, u(s,X)\, \mathrm{d}X.$$

Notice that $g(0) = f(t)$. We have, reminding that K satisfies the heat equation (parameters have been "forgotten" for better clarity):

$$\begin{aligned} g'(s) &= \int_{\mathbb{R}^N} -\Delta K \cdot u + \left(\Delta u + u^{1+p}\right) \cdot K\, \mathrm{d}X \\ &= \underbrace{\int_{\mathbb{R}^N} -\Delta K \cdot u + \Delta u \cdot K\, \mathrm{d}X}_{=0 \text{ thanks to an IBP}} + \int_{\mathbb{R}^N} K u^{1+p}\, \mathrm{d}X \\ &= \int_{\mathbb{R}^N} K u^{1+p}\, \mathrm{d}X \end{aligned}$$

we use then the Jensen inequality thanks to the convexity of $(\bullet)^{1+p}$ and the fact that $\|K(t,\bullet)\|_{L^1} = 1$; thus

$$\begin{aligned} g'(s) &\geq \left(\int_{\mathbb{R}^N} K u\right)^{1+p} \\ &= (g(s))^{1+p}. \end{aligned}$$

Therefore g satisfies the ordinary differential inequation $g' \geq g^{1+p}$. By separating the variables, one reaches

$$\frac{1}{g^p(0)} - \frac{1}{g^p(t)} \geq pt.$$

Then, since g is positive, we get

$$\frac{1}{g^p(0)} \geq pt$$

whence one achieves the upper bound of f,

$$f(t) = g(0) \leq \frac{p^{1/p}}{t^{1/p}} = \frac{\widetilde{C}}{t^{1/p}}. \qquad (\mathbf{I.13})$$

As a conclusion, by gathering (I.12) and (I.13), we find the following bounds for f:

$$\frac{C}{t^{N/2}} \leq f(t) \leq \frac{\widetilde{C}}{t^{1/p}}. \qquad (\mathbf{I.14})$$

However, since $p < 2/N$, one has $1/p > N/2$ and so $\widetilde{C}/t^{1/p}$ decays faster than $C/t^{N/2}$. Then the inequation (I.14) cannot be verified when t becomes large as you can see on the figure (*F18*).

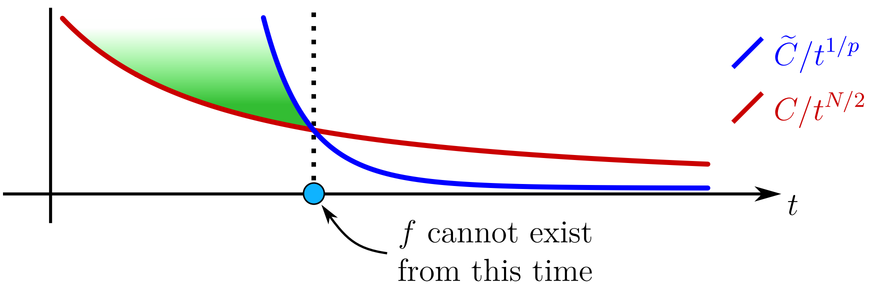


Figure F18 – *Illustration of the contradiction given by (I.14): f should be both above the red curve and bellow the blue one; it is the case for the small values of t but not for the large one.*

Thereby, the assumption under which the solution u is global (this hypothesis has allowed us to create the function f) is wrong ans so u blows up in a finite time; that's what we were aiming for. □

I.5.2 Hair Trigger Effect *versus* Extinction

Coming back to the whole problem (I.10) which we recall here,

$$\begin{cases} \partial_t u = d\Delta u + r u^{1+p}(1-u) & (t, X) \in (0;\infty) \times \mathbb{R}^N \\ u(0, X) = u_0(X) & X \in \mathbb{R}^N, \end{cases}$$

one knows, thanks to the comparison principle, that all the solutions starting from an initial datum between the both stationary states 0 and 1 are global and remain between these bounds. The question of the lifetime solution is thus settled. We are now asking about the long time behaviour of the solutions. When we have presented the Fisher-KPP equations (see page 24), and especially the logistic one whom we recall the associated Cauchy problem bellow,

$$\begin{cases} \partial_t u = d\Delta u + ru^{1+0}(1-u) & (t,X) \in (0;\infty) \times \mathbb{R}^N \\ u(0,X) = u_0(X) & X \in \mathbb{R}^N, \end{cases}$$

there was Hair Trigger Effect, *i.e.* a systematic space invasion toward the carrying capacity $u_\infty \equiv 1$ locally uniformly in space, for each non-negative, smaller that 1 and non-zero initial datum. Now we are inducing an Allee effect by changing the reaction function $ru^{1+0}(1-u)$ into $ru^{1+p}(1-u)$, one can ask whether this new R-D Cauchy problem generate

- an Hair Trigger Effect, due to the vicinity of $ru^{1+p}(1-u)$ from the logistic reaction in the case where p is near from 0,
- an extinction of the population, due to the hardness of the Allee effect in the case where p is far from 0.

Actually, the study of the Fujita's blow up we have done before is a good track to try to answer the question. Indeed, the Fujita's theorem give us a critical exponent p_F which splits systematic and non-systematic blow-up of the solutions starting from some non-negative and non-zero initial datums.

- For a soft Allee effect ($p \le p_F$), there is an uplift of all the solutions toward the blow up. Therefore one can imagine in this case that there is Hair Trigger Effect once we add the barrier term $(1-u)$.
- For an hard Allee effect ($p > p_F$), there exists some initial datums for which the population becomes extinct and so one can think it is the same when one adds $(1-u)$ in the reaction term.

Both of these assumptions have been proved by Aronson and Weinberger in 1978 [2]

Theorem 23 (Aronson-Weinberger) (HTE *vs.* extinction) (1978)
Consider again the Fujita's exponent $p_F := 2/N$.

Soft Allee effect If $0 < p \le p_F$, then for all non-negative, smaller than 1 and non-zero initial datum there is Hair Trigger Effect for the solution of (I.10); that is, the solution tends toward $u_\infty \equiv 1$ locally uniformly in space as t tends to $+\infty$.

Hard Allee effect If $p > p_F$, then there exists some "small enough" non-negative initial datums for which the solution of (I.10) shall become extinct; that is, the solution tends toward $u_\infty \equiv 0$ locally uniformly in space as t tends to $+\infty$.

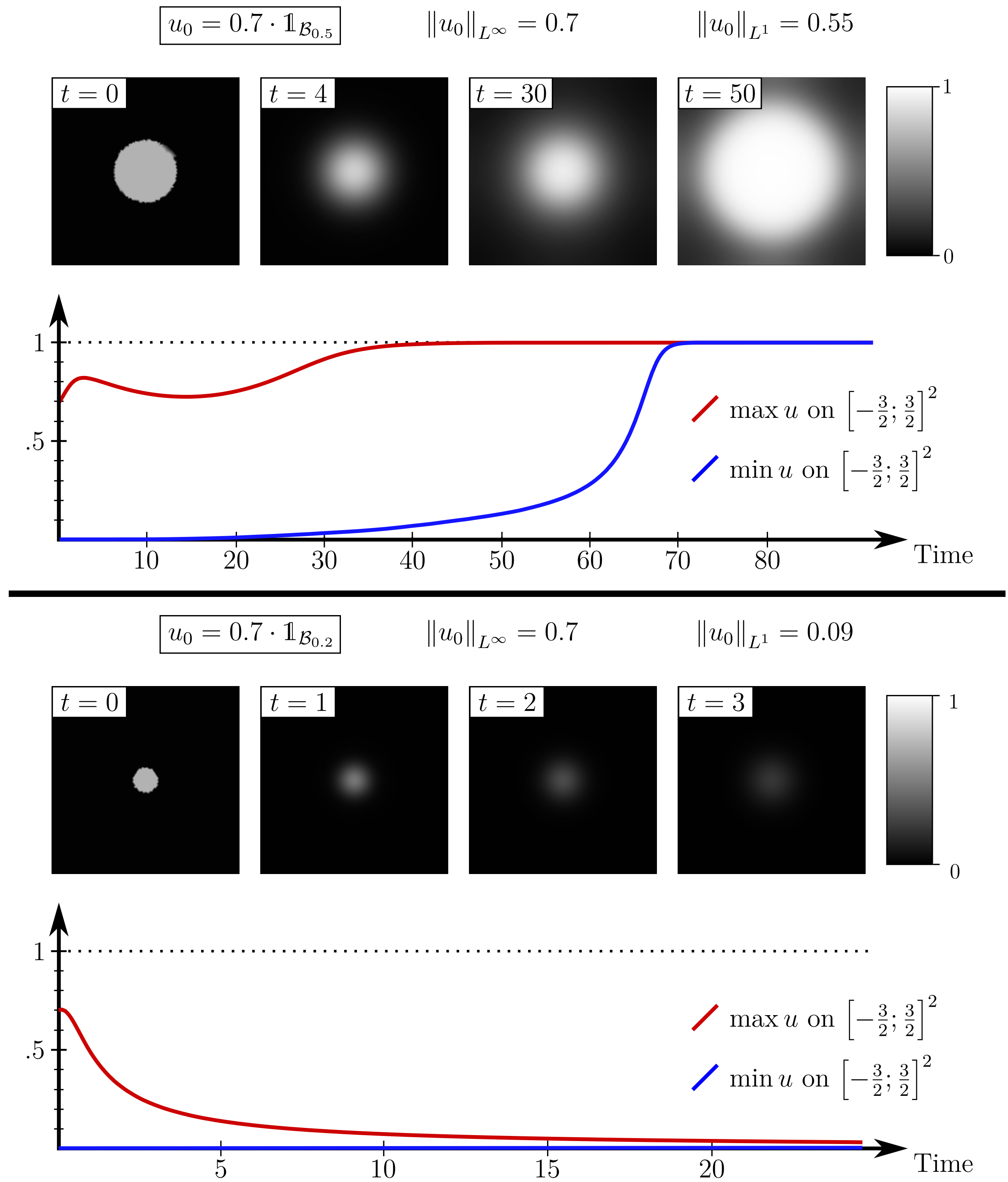


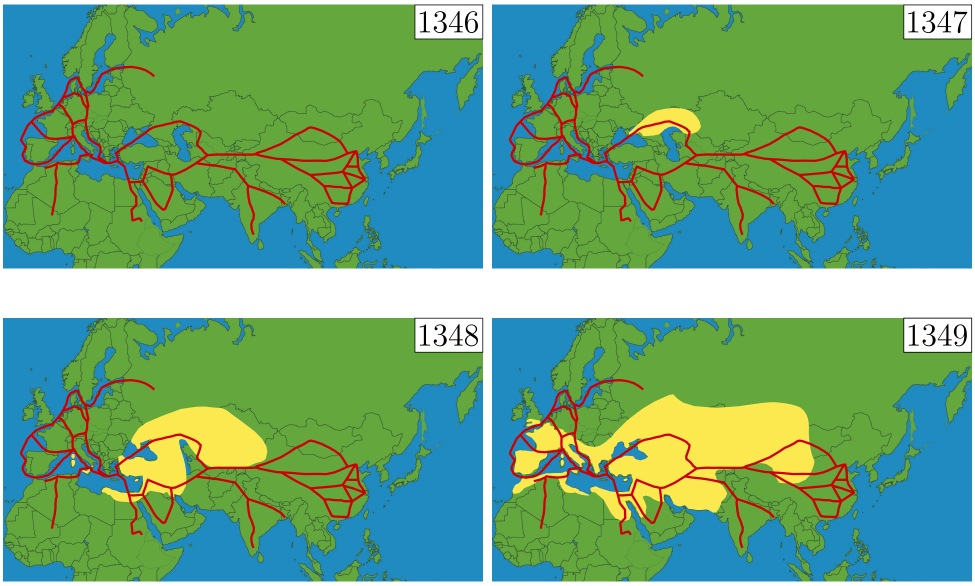
Figure F19 – *Experiment results about the "hard Allee effect part" of the Aronson-Weinberger theorem. Because $N = 2$, $p_F = 1$. We had thus taken $p = 3$. One compares here the long time evolution for two given initial datums. Like in figure (F12) (page 26), there are on the top of each case four snapshots of the solution viewed from the window $\left[-\frac{3}{2}; \frac{3}{2}\right]^2$, and on the bottom the local* min *and* max *of this solution in the same window. In the first case, u_0 is "large enough" for the Hair Trigger Effect to happen, but in the second one u_0 is "too small" to allow the population to persist.*

PART II

Fisher-KPP Reaction-Diffusion Equations on the Field-Road space $\mathbb{R}^2_+$

In this whole part we shall present and discuss on the paper "*The influence of a line with fast diffusion on Fisher-KPP propagation*" published in 2018 by Berestycki *et al.* [4]. In the sequence, the terms "the authors" and "the article" refer then to the latter. One reaches to model here a non-homogeneous diffusion in space and more precisely some significant increase of the diffusion along certain axes. This will is motivated by a number of ecological and biological observations whom we give some bellow.

- The first example given is the one of the "*Black death*" *plague* which occurs in the middle of the 14th century. The front of this epidemic has actually been accelerated because of the so called *silk road* which was a trade road connecting many ports and cities in Asia, Africa and Europe. One can refer for instance to [17] in order to get further details.



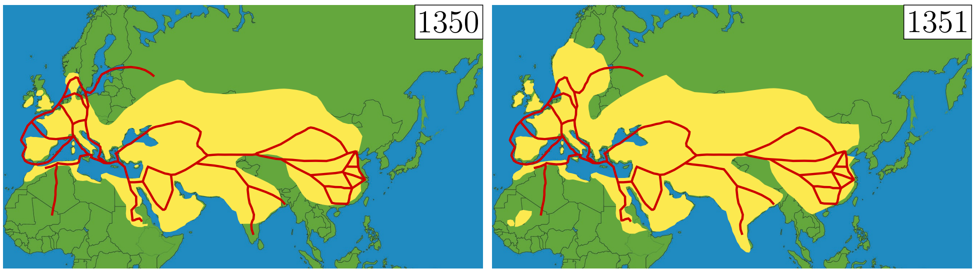


Figure F20 – *The Black death plague's spread from 1346 to 1351. One clearly sees that the epidemic follows the silk road drawn in red.*

- We continue in the same vein by talking about a more contemporary problem: COVID epidemic. It has been shown that the virus has a faster circulation along highways and transportation infrastructures [8].
- We may also say a few words about the processionary caterpillar of the pine tree and the "Asian tiger mosquito" which both have been found at some place that we didn't expect; this being again due to the human means of transport. For this example one refers the reader to [14] for the processionary and to [3] for the mosquito.
- In [9] one observes that rivers allow faster spreading of plants pathologies.
- And the last illustration we shall present here is the case of western Canadian wolfs. Using a GPS tracking system, McKenzie *et al.* have reported in [11] that wolfs use the *seismic lines* ([a]) in order to move faster: "wolves moved up to 2.8 times faster on linear features than in the forest". This phenomenon disrupts the local ecology equilibrium because it increase the chance of wolfs to find some preys – one especially thinks about the caribou which constitute there an endangered species.

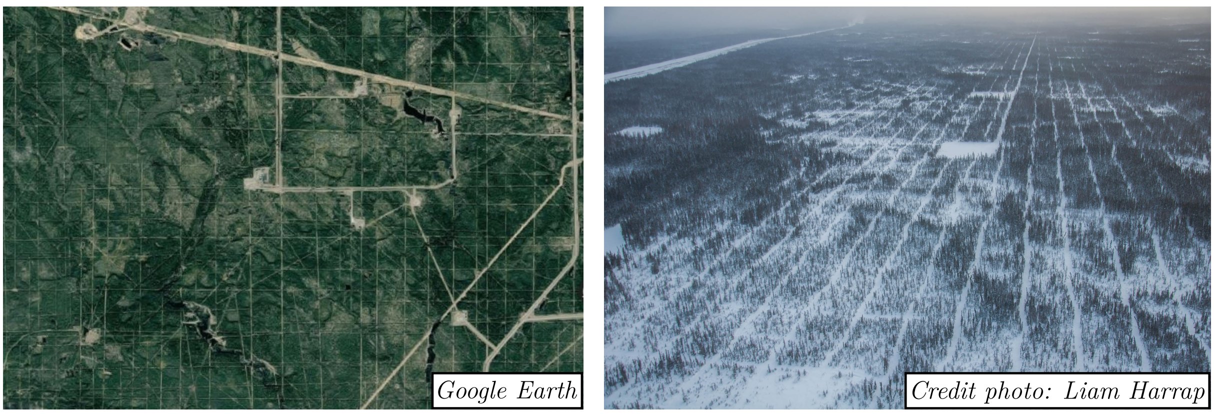


Figure F21 – *On left: satellite view of seismic lines near Zama City, Canada. On right: seismic lines viewed from plane.*

[a] These are clear, straight and man-made lines which cross the forest used to detect natural oil or gas reserve. They are large of about 5m and long about a few kilometres, as one sees on figure (*F21*).

Figure F22 – *Photographs of a Canadian caribou (Rangifer tarandus) and Canadian wolf (Canis lupus).*

II.1 Presentation of the Field-Road model

One takes place in a 2-dimensional space [b]. In order to model some phenomenon of diffusion acceleration on a line, which may allow us to describe the biological facts we mentioned above, the authors propose to work on the half-plane domain $\mathbb{R}^2_+ := \mathbb{R} \times \mathbb{R}_+$ [c] which we shall call a *Field-Road space*. Let us take a *single* population living then on that place; one distinguish there two disjoint locations:

$\boxed{\text{The Field}}$ is the subset $\mathcal{F} := \mathbb{R} \times \mathbb{R}^*_+$. Let $v : [0; \infty) \times \mathcal{F} \to \mathbb{R}$ denote the population density in $\mathcal{F}$, we assume a reaction-diffusion equation acts on v, with

- a small coefficient d for the diffusion, and
- a reaction function f of Fisher-KPP type.

This means that individuals can reproduce and slowly move on the Field.

$\boxed{\text{The Road}}$ is the subset $\mathcal{R} := \mathbb{R} \times \{0\}$. Let $u : [0; \infty) \times \mathcal{R} \to \mathbb{R}$ denote the population density in $\mathcal{R}$, we suppose this time that only a diffusion equation acts on u, with

- a large diffusion coefficient D.

This means that individuals can only move fast on the Road.

Remarks.

1. The population we study is thus described by a couple of functions (u, v).

[b] Note one actually can work in N dimensions ($N \geq 2$) like it is done in the paper. Because the arguments in the N-dimensional cases are the same as those in the 2-dimensional one, we chose $N = 2$ for better readability.

[c] Take the half space $\mathbb{R}^{N-1} \times \mathbb{R}_+$ for the N-dimensional cases.

2. Note that $\mathcal{R}$ is the frontier of $\mathcal{F}$ and that $\mathbb{R}^2_+ = \mathcal{R} \cup \mathcal{F}$.

For the time being, the both functions u and v are not related. We now define some laws of individuals exchanges between the Field and the Road. At every moment,

- a certain proportion of the Field individuals hitting the Road get in there and the others rebound in the Field;
- some other proportion of the Road individuals leave the Road to get into the Field and the others stay on the Road.

Under the assumptions we mentioned above, one can now draw up the following PDE/Boundary conditions system:

$$\begin{cases} \partial_t u - D\partial_{xx} u = \nu v\left(t,x,0\right) - \mu u & \left(t,x,0\right) \in (0;\infty) \times \mathcal{R} \\ \partial_t v - d\Delta v = f\left(v\right) & \left(t,x,y\right) \in (0;\infty) \times \mathcal{F} \\ -d\partial_y v\left(t,x,0\right) = \mu u\left(t,x\right) - \nu v\left(t,x,0\right) & \left(t,x,0\right) \in (0;\infty) \times \mathcal{R}, \end{cases} \tag{II.1}$$

where μ and ν are real positive numbers related to the exchanges proportions between Field and Road.

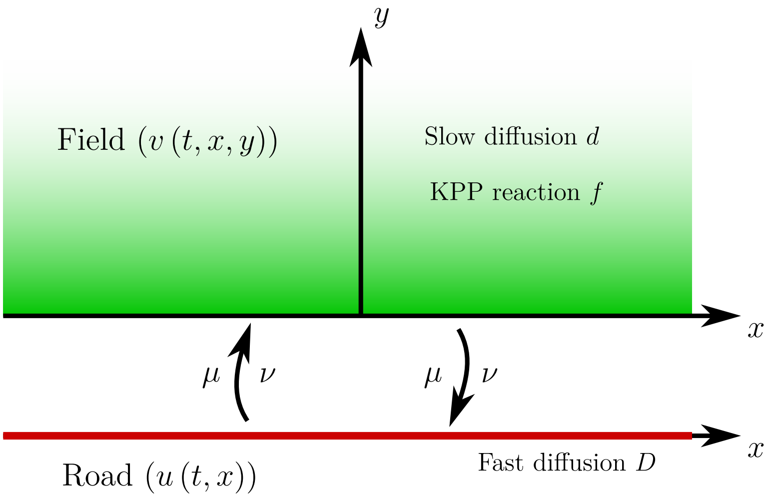


Figure F23 – *Sketch of the Field-Road space in $\mathbb{R}^2$. Note the both x-axis should be at the same place but we separated them for better readability. Note the frontier of the Field plays an important role in the model as an exchange interface between the Field and the Road.*

The first thing we are going to do, is notice that taking $\nu = 1$ does not entail any loss of generality. Indeed if we rescale the time by a factor $1/\nu$, that is by taking $\left(\widetilde{u}\left(t,x\right), \widetilde{v}\left(t,x,y\right)\right) := \left(u\left(t/\nu,x\right), v\left(t/\nu,x,y\right)\right)$, we obtain the equivalent to system (II.1)

$$\begin{cases} \partial_t \widetilde{u} - \frac{D}{\nu}\partial_{xx}\widetilde{u} = \widetilde{v}\left(t,x,0\right) - \frac{\mu}{\nu}\widetilde{u} & \left(t,x,0\right) \in (0;\infty) \times \mathcal{R} \\ \partial_t \widetilde{v} - \frac{d}{\nu}\Delta\widetilde{v} = \frac{f\left(\widetilde{v}\right)}{\nu} & \left(t,x,y\right) \in (0;\infty) \times \mathcal{F} \\ -\frac{d}{\nu}\partial_y \widetilde{v}\left(t,x,0\right) = \frac{\mu}{\nu}\widetilde{u}\left(t,x\right) - \widetilde{v}\left(t,x,0\right) & \left(t,x,0\right) \in (0;\infty) \times \mathcal{R}, \end{cases}$$

whence by recalling

- $\mu := \mu/\nu$,
- $D := D/\nu$,
- $d := d/\nu$,
- $f := f/\nu$,

one can get rid of ν which implicitly take place in the new parameters. Thereby, we now work on

$$\begin{cases} \partial_t u - D\partial_{xx}u = v\left(t,x,0\right) - \mu u & (t,x,0) \in (0;\infty) \times \mathcal{R} \\ \partial_t v - d\Delta v = f\left(v\right) & (t,x,y) \in (0;\infty) \times \mathcal{F} \\ -d\partial_y v\left(t,x,0\right) = \mu u\left(t,x\right) - v\left(t,x,0\right) & (t,x,0) \in (0;\infty) \times \mathcal{R}. \end{cases} \qquad \textbf{(II.2)}$$

II.2 Population mass conservation

As mentioned in first part (see page 16), a realistic R-D model should preserve the quantity of individuals whether there is no reaction – *i.e.* $f \equiv 0$. We had already shown it in the $\mathbb{R}^N$ case in first part, but we have to do it again here because it is *a priori* not sure that *all* individuals leaving the Field get in the Road and *vice versa*; otherwise said, some individuals could be "lost" during the exchanges, so we have to check that there is no "individuals leakage" in the model. Fortunately, everything is going fine:

Proposition 24 (Population mass conservation on the Field-Road)
Assume $f \equiv 0$ in problem (II.2). If (u, v) is a solution of this problem such that the both functions u and v are non-negatives and decay in space, for all $t \geq 0$, faster than some exponential functions ([a]), then the total population described by the amount

$$\mathcal{M}\left(t\right) := \left\|u\left(t,\bullet\right)\right\|_{L^1(\mathbb{R})} + \left\|v\left(t,\bullet,\bullet\right)\right\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)}$$

remains constant with respect to the time. In an equivalent way, we have $\mathcal{M}\left(t\right) = \mathcal{M}\left(0\right)$ for all $t > 0$.

[a] It is actually sufficient, thanks to parabolic estimates, to suppose this property of exponential decadence true at the initial time to recover it for all $t \geq 0$.

Proof (Proposition 24)
For $T > 0$, we assess the population variation between the instants 0 and T, on the Road from one part and in the Field from another part. By summing these both quantities, we reach the total population variation in the whole space. Then one finds out that the population which has been disappeared on the Road until the instant T has actually been gained in the Field and *vice versa*. In other words, the exchanges between the Field and the Road compensate each other and we finally get the expected result.

Population variation on the Road between times 0 and T

$$\begin{aligned}\|u(T,\bullet)\|_{L^1(\mathbb{R})} - \|u(0,\bullet)\|_{L^1(\mathbb{R})} &= \int_{-\infty}^{+\infty} u(T,x) - u(0,x)\,\mathrm{d}x \\ &= \int_{-\infty}^{+\infty}\int_0^T \partial_t u(t,x)\,\mathrm{d}t\,\mathrm{d}x\end{aligned}$$

by using Fubini's theorem,

$$= \int_0^T \int_{-\infty}^{+\infty} \partial_t u(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

by using first equation in (II.2),

$$\begin{aligned}&= \int_0^T \int_{-\infty}^{+\infty} v(t,x,0) - \mu u(t,x) + D\partial_{xx} u(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^T \Big[\int_{-\infty}^{+\infty} v(t,x,0) - \mu u(t,x)\,\mathrm{d}x \\ &\qquad + D\underbrace{\left(\partial_x u(t,+\infty) - \partial_x u(t,-\infty)\right)}_{=0 \text{ thanks to the fast decay of } u}\Big]\mathrm{d}t\end{aligned}$$

$$\|u(T,\bullet)\|_{L^1(\mathbb{R})} - \|u(0,\bullet)\|_{L^1(\mathbb{R})} = \int_0^T \int_{-\infty}^{+\infty} v(t,x,0) - \mu u(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Population variation on the Field between times 0 and T

$$\begin{aligned}\|v(T,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} - \|v(0,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} &= \int_0^T \int_{\mathbb{R}\times\mathbb{R}_+^*} \partial_t v(t,x,y)\,\mathrm{d}x\mathrm{d}y\,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}\times\mathbb{R}_+^*} d\Delta v(t,x,y) + \underbrace{f(v(t,x,y))}_{=0}\mathrm{d}x\mathrm{d}y\,\mathrm{d}t\end{aligned}$$

integrate then the latter by part using Green's formula,

$$\|v(T,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} - \|v(0,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} = \int_0^T \int_{-\infty}^{+\infty} -d\partial_y v(t,x,0)\,\mathrm{d}x\,\mathrm{d}t$$

finaly by using third equation in (II.2),

$$\|v(T,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} - \|v(0,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} = \int_0^T \int_{-\infty}^{+\infty} \mu u(t,x) - v(t,x,0)\,\mathrm{d}x\,\mathrm{d}t.$$

We can now observe that the population variation in the Field is opposed to the one on the Road. Thereby, by summing, there comes

$$\|u(T,\bullet)\|_{L^1(\mathbb{R})} - \|u(0,\bullet)\|_{L^1(\mathbb{R})} + \|v(T,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} - \|v(0,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)} = 0,$$

whence

$$\|u(T,\bullet)\|_{L^1(\mathbb{R})}+\|v(T,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)}=\|u(0,\bullet)\|_{L^1(\mathbb{R})}\,\|v(0,\bullet,\bullet)\|_{L^1\left(\mathbb{R}\times\mathbb{R}_+^*\right)},$$

that is $\mathcal{M}(T)=\mathcal{M}(0)$. □

II.3 A few words about the μ parameter

One discuss here about the signification of the μ parameter that is related to migrations between the Field and the Road. This amount μ should not be interpreted as a *migratory intensity* but rather as an *attractive equilibrium ratio* between population mass on the frontier of the Field and on the Road. This equilibrium yields for

$$\frac{\|v(t,\bullet,0)\|_{L^1(\mathbb{R})}}{\|u(t,\bullet)\|_{L^1(\mathbb{R})}}=\mu,$$

and whether that equality may not achieved, $\|v(t,\bullet,0)\|_{L^1}$ and $\|u(t,\bullet)\|_{L^1}$ evolve in order to get their ratio closer to μ.

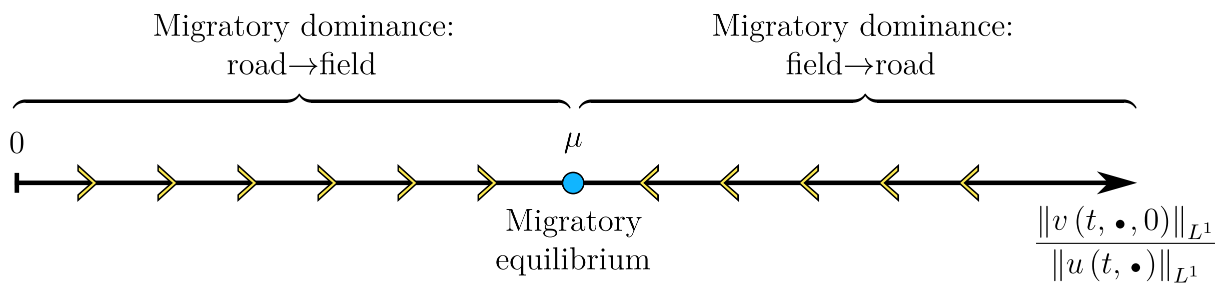


Figure F24 – *Illustration of the migratory dominances between the Field and the Road depending on the position of $\frac{\|v(t,\cdot,0)\|_{L^1}}{\|u(t,\cdot)\|_{L^1}}$ with respect to μ.*

Therefore population mass evolve toward the ratio Road/Field-frontier "one for μ" and we thus have the three following possible cases:

- $\mu=1$ means a neutral equilibrium "one for one",
- $\mu>1$ means an equilibrium promoting the Field,
- $\mu<1$ means an equilibrium in favour of the Road.

All which have been clamed from the beginning of section II.3 is now going to be verified. Let t be a real non-negative number, we compute at the instant t the time derivative of the population mass on the Road, that also is the population flow from the Field to the Road because all variations of the population mass on the Road is due to the migrations between the Field and the Road.

$$\begin{aligned}\partial_t\left(\|u(t,\bullet)\|_{L^1(\mathbb{R})}\right) &= \partial_t\left(\int_{-\infty}^{+\infty} u(t,x)\,\mathrm{d}x\right)\\ &= \int_{-\infty}^{+\infty} \partial_t u(t,x)\,\mathrm{d}x\end{aligned}$$

using then first line of (II.2),

$$\begin{aligned}\partial_t\left(\|u(t,\bullet)\|_{L^1(\mathbb{R})}\right) &= \int_{-\infty}^{+\infty} v(t,x,0)-\mu u(t,x)+D\partial_{xx}u(t,x)\,\mathrm{d}x\\ &= \int_{-\infty}^{+\infty} v(t,x,0)-\mu u(t,x)\,\mathrm{d}x + D\underbrace{\left(\partial_x u(t,+\infty)-\partial_x u(t,-\infty)\right)}_{=0\text{ under assumptions of proposition 24.}}\\ &= \|v(t,\bullet,0)\|_{L^1(\mathbb{R})}-\mu\|u(t,\bullet)\|_{L^1(\mathbb{R})}.\end{aligned}$$

Whence the exchanges of individuals are lead by the following:

- If $\frac{\|v(t,\cdot,0)\|_{L^1}}{\|u(t,\cdot)\|_{L^1}}=\mu$ then the individuals flow from Field to Road is zeros that is there is no migration between the Field and the Road.
- If $\frac{\|v(t,\cdot,0)\|_{L^1}}{\|u(t,\cdot)\|_{L^1}}>\mu$ then the individuals flow from Field to Road is positive that is the individuals mainly migrate from the Field to the Road.
- If $\frac{\|v(t,\cdot,0)\|_{L^1}}{\|u(t,\cdot)\|_{L^1}}<\mu$ then the individuals flow from Field to Road is negative that is the individuals mainly migrate from the Road to the Field.

II.4 Cauchy problem: Existence, uniqueness, CP

One deals in this section with the problem (II.2) provided with the initial datum

$$\begin{cases} u(0,\bullet)=u_0 & \text{in } \mathbb{R},\\ v(0,\bullet,\bullet)=v_0 & \text{in } \mathbb{R}\times\mathbb{R}_+^*,\end{cases} \qquad \textbf{(II.3)}$$

such that u_0 and v_0 are each non-negative, continuous and bounded by 1.

Theorem 25 (Well-posedness of the R-D Cauchy problem)
Problem (II.2) combined with the initial datum (II.3) admits a unique solution (u,v) which share the same properties as the one we set on (u_0,v_0), namely u and v are each non-negative, continuous and bounded by 1.

Remark. Note we do not demand that the extension of v_0 by u_0 on the Road is continuous. Therefore, v extended by u on the Road has no reason to be continuous either.

We shall admit in the sequence the existence part of theorem 25. The uniqueness part derives from the comparison principle which we will announce after these few definitions:

Definition 26 (Sub-solution)
Let $\underline{u} \in \mathcal{C}^{1,2}([0;T) \times \mathbb{R}, \mathbb{R})$ and $\underline{v} \in \mathcal{C}^{1,2}\left([0;T) \times \mathbb{R} \times \mathbb{R}_+^*, \mathbb{R}\right)$, on says that $(\underline{u}, \underline{v})$ is a *sub-solution* for the problem (II.2) if

$$\begin{cases} \partial_t \underline{u} - D\partial_{xx}\underline{u} \leq \underline{v}(t,x,0) - \mu \underline{u} & (t,x,0) \in (0;\infty) \times \mathcal{R} \\ \partial_t \underline{v} - d\Delta \underline{v} \leq f(\underline{v}) & (t,x,y) \in (0;\infty) \times \mathcal{F} \\ -d\partial_y \underline{v}(t,x,0) \leq \mu \underline{u}(t,x) - \underline{v}(t,x,0) & (t,x,0) \in (0;\infty) \times \mathcal{R}. \end{cases}$$

Definition 27 (Super-solution)
Let $\overline{u} \in \mathcal{C}^{1,2}([0;T) \times \mathbb{R}, \mathbb{R})$ and $\overline{v} \in \mathcal{C}^{1,2}\left([0;T) \times \mathbb{R} \times \mathbb{R}_+^*, \mathbb{R}\right)$, on says that $(\overline{u}, \overline{v})$ is a *super-solution* for the problem (II.2) if

$$\begin{cases} \partial_t \overline{u} - D\partial_{xx}\overline{u} \geq \overline{v}(t,x,0) - \mu \overline{u} & (t,x,0) \in (0;\infty) \times \mathcal{R} \\ \partial_t \overline{v} - d\Delta \overline{v} \geq f(\overline{v}) & (t,x,y) \in (0;\infty) \times \mathcal{F} \\ -d\partial_y \overline{v}(t,x,0) \geq \mu \overline{u}(t,x) - \overline{v}(t,x,0) & (t,x,0) \in (0;\infty) \times \mathcal{R}. \end{cases}$$

Theorem 28 (Parabolic non-linear comparison principle)
Let us consider

- $(\underline{u}, \underline{v})$ a sub-solution for (II.2) bounded from above,
- $(\overline{u}, \overline{v})$ a super-solution for (II.2) bounded from bellow,

satisfying $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$, when $t = 0$, then

- either $(\underline{u}, \underline{v}) < (\overline{u}, \overline{v})$ for all $t \in (0; +\infty)$,
- or there exists some $T > 0$ such that $(\underline{u}, \underline{v}) = (\overline{u}, \overline{v})$ in $[0;T]$.

One can by now use the comparison principle to build the demonstration of the uniqueness part of theorem 25.

Proof (Theorem 25)(Well-posedness)(Uniqueness part)
Let (u,v) and $(\tilde{u}, \tilde{v})$ be two solutions of (II.2) starting from the same initial datum (u_0, v_0). This proof shall be completed when it will have been shown that $(u,v) \equiv (\tilde{u}, \tilde{v})$. To do this, assume by contradiction that it is not the case. Then the set

$$\mathscr{T} := \{T \geq 0 \ / \ (u,v) = (\tilde{u}, \tilde{v}) \text{ for all } t \in [0;T]\}$$

is bounded (due to the absurd hypothesis) and not empty (0 is in). We are then allowed to set

$$T_{\max} := \sup \mathscr{T} < +\infty.$$

We show now two things about $T_{\max}$:

$\boxed{\mathbf{1}}$ $T_{\max}$ is positive.

Indeed, assume that $T_{\max} = 0$, by applying the comparison principle with $(\underline{u}, \underline{v}) = (u, v)$ and $(\overline{u}, \overline{v}) = (\tilde{u}, \tilde{v})$, the only possibility is that $(u, v) < (\tilde{u}, \tilde{v})$ for all time. Repeat that by switching $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ we get $(u, v) > (\tilde{u}, \tilde{v})$ for all time which is obviously conflicting with the previous inequality. Thereby, $T_{\max} > 0$.

$\boxed{\mathbf{2}}$ $(u, v) = (\tilde{u}, \tilde{v})$ at $T_{\max}$.

To see that, take a maximizing sequence $(T_n)_{n \in \mathbb{N}} \subset \mathscr{T}$ that converges up to $T_{\max}$. Because for all $n \in \mathbb{N}$, $(u, v) - (\tilde{u}, \tilde{v}) = (0, 0)$ at T_n and $(u, v) - (\tilde{u}, \tilde{v})$ is continuous with respect to the time, one finds out that $(u, v) = (\tilde{u}, \tilde{v})$ at $T_{\max}$.

It is now easy to conclude: like $\boxed{1}$ has been proved, by applying twice the comparison principle for (u, v) and $(\tilde{u}, \tilde{v})$ starting at time $T_{\max}$, the maximality of $T_{\max}$ forces us to conclude both $(u, v) < (\tilde{u}, \tilde{v})$ and $(u, v) > (\tilde{u}, \tilde{v})$ for all time larger that $T_{\max}$. That's absurd. $\square$

Theorem 28 can be extended to "generalised sub- and super-solutions" which are respectively defined by the sup of sub-solutions and the inf of super-solutions by the following way.

Proposition 29 (Generalised parabolic non-linear comparison principle) One considers two parabolic and open domains $E \subset (0; \infty) \times \mathcal{R}$ and $F \subset (0; \infty) \times \mathcal{F}$, and (u_1, v_1) and (u_2, v_2) two sub-solutions for (II.2) bounded from above and satisfying

$$\begin{cases} u_1 \le u_2 & \text{on } (\partial E) \cap ((0; \infty) \times \mathcal{R}) \\ v_1 \le v_2 & \text{on } (\partial F) \cap ((0; \infty) \times \mathcal{F}) . \end{cases}$$

If the functions $\underline{u}$ and $\underline{v}$ defined by

$$\underline{u}(t, x) := \begin{cases} \max(u_1(t, x), u_2(t, x)) & \text{if } (t, x) \in \overline{E} \\ u_2(t, x) & \text{otherwise,} \end{cases}$$

$$\underline{v}(t, x, y) := \begin{cases} \max(v_1(t, x, y), v_2(t, x, y)) & \text{if } (t, x, y) \in \overline{F} \\ v_2(t, x, y) & \text{otherwise,} \end{cases}$$

satisfy

$$\begin{aligned} [\underline{u}(t, x) > u_2(t, x)] &\Rightarrow [\underline{v}(t, x, 0) \le v_1(t, x, 0)] \\ [\underline{v}(t, x, 0) > v_2(t, x, 0)] &\Rightarrow [\underline{u}(t, x) \le u_1(t, x)] , \end{aligned} \tag{II.4}$$

then for all bounded from bellow super-solution $(\overline{u}, \overline{v})$ for (II.2) such that $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$ at time $t = 0$, that inequality spreads on all positive time, that is $(\underline{u}, \underline{v}) \le (\overline{u}, \overline{v})$ for all $t \ge 0$.

Remark. For parabolic equations in $\mathbb{R}^N$ the comparison principle always holds for generalised sub-solution (*i.e.* the sup of sub-solutions) and super-solutions (*i.e.* the inf of super-solutions). For the Field-Road model, one actually needs the extra assumption (II.4) to get that true. Note that whether $(\underline{u}, \underline{v})$ is a compactly supported sub-solution, the zero-extension of $(\underline{u}, \underline{v})$ satisfies assumption (II.4) with $(u_1, v_1) := (\underline{u}, \underline{v})$ and $(u_2, v_2) :\equiv (0, 0)$.

II.5 Long time behaviour

We are interested in this section in the future of solutions of the Cauchy problem (II.2)–(II.3) in long time. To deal with this, we are looking for stationary solutions of system (II.2); we work so on the following timeless problem

$$\begin{cases} -D\partial_{xx}U = V(x,0) - \mu U & (x,0) \in \mathcal{R} \\ -d\Delta V = f(V) & (x,y) \in \mathcal{F} \\ -d\partial_y V(x,0) = \mu U(x) - V(x,0) & (x,0) \in \mathcal{R}. \end{cases} \qquad \textbf{(II.5)}$$

Proposition 30 (Stationary solutions)

There are only two bounded and non-negative solutions of (II.5) which are:

$$(U, V) \equiv (0, 0) \qquad \text{and} \qquad (U, V) \equiv \left(\frac{1}{\mu}, 1\right).$$

Remark. According to section II.3, it is not surprising to see that stationary solutions of system (II.2) satisfy both their mass ratio Field-frontier/Road is at the equilibrium μ.

Lemma 31

Assume that (U, V) is a bounded positive solution of (II.5), then for all $r > 0$,

$$\inf_{\mathbb{R}\times[r;+\infty)} V > 0.$$

Proof (Lemma 31)

As in the Hair Trigger Effect proof (see page 27), we denote by (λ, φ) the couple principal-(eigenvalue/eigenfunction) of $-d\Delta$ with Dirichlet boundary condition in $\mathcal{B}_R$ as defined in section 2 of the Toolbox part page 118. Take $R > 0$ large enough so that

$$\lambda < \frac{f'(0)}{2d}.$$

Then, by letting $\varepsilon > 0$,

$$\begin{aligned}-d\Delta(\varepsilon\varphi) - f(\varepsilon\varphi) &= d\varepsilon\lambda\varphi - f(\varepsilon\varphi)\\ &= \varepsilon\varphi\left(d\lambda - \frac{f(\varepsilon\varphi)}{\varepsilon\varphi}\right)\\ &< \varepsilon\varphi\left(\frac{f'(0)}{2} - \frac{f(\varepsilon\varphi - f(0))}{\varepsilon\varphi - 0}\right).\end{aligned}$$

Yet, $\frac{f'(0)}{2} - \frac{f(\varepsilon\varphi - f(0))}{\varepsilon\varphi - 0}$ tends to $-\frac{f'(0)}{2} < 0$ as ε tends to 0; whence for ε_0 close enough to 0, we have for all $0 < \varepsilon \leq \varepsilon_0$,

$$-d\Delta(\varepsilon\varphi) - f(\varepsilon\varphi) < \underbrace{\varepsilon\varphi}_{>0}\underbrace{\left(\frac{f'(0)}{2} - \frac{f(\varepsilon\varphi - f(0))}{\varepsilon\varphi - 0}\right)}_{<0} < 0. \qquad \textbf{(II.6)}$$

Claim. For all $(x_0, y_0) \in \mathbb{R} \times (R; +\infty)$ and for all $(x, y) \in \mathcal{B}_R(x_0, y_0)$,

$$V(x, y) \geq \varepsilon_0\varphi(x - x_0, y - y_0).$$

Let's prove that claim: we assume by contradiction that $V < \varepsilon_0\varphi(\bullet - x_0, \bullet - y_0)$ somewhere in $\mathcal{B}_R(x_0, y_0)$. Then as V is positive in $\mathcal{F}$, by taking ε smaller, one may have $\varepsilon_0\varphi(\bullet - x_0, \bullet - y_0) < V$ in $\mathcal{B}_R(x_0, y_0)$. Hence the set

$$\mathcal{E} := \{\varepsilon > 0 \ / \ \forall (x, y) \in \mathcal{B}_R(x_0, y_0), \varepsilon\varphi(x - x_0, y - y_0) < V(x, y)\}$$

is non-empty and bounded from above by ε_0 thanks to the absurd hypothesis. So one can take $\varepsilon^* := \sup \varepsilon < \varepsilon_0$. Let us call (x^*, y^*) the contact point between V and $\varepsilon^*\varphi(\bullet - x_0, \bullet - y_0)$ which cannot belong to the frontier $\partial\mathcal{B}_R(x_0, y_0)$ because φ is zero on that set. In such a contact point, $V - \varepsilon^*\varphi(\bullet - x_0, \bullet - y_0)$ achieves a not-on-the-frontier local minimum therefore

$$d\Delta(V(x^*, y^*) - \varepsilon^*\varphi(x^* - x_0, y^* - y_0)) \geq 0. \qquad \textbf{(II.7)}$$

Furthermore,

$$d\Delta(V - \varepsilon^*\varphi(\bullet - x_0, \bullet - y_0)) = d\Delta V - d\Delta(\varepsilon^*\varphi(\bullet - x_0, \bullet - y_0))$$

and we use (II.6) because $\varepsilon^* \leq \varepsilon_0$ (thanks to the absurd hypothesis),

$$d\Delta(V - \varepsilon^*\varphi(\bullet - x_0, \bullet - y_0)) < f(\varepsilon^*\varphi(\bullet - x_0, \bullet - y_0)) - f(V).$$

Finally, by recalling that V and $\varphi(\bullet - x_0, \bullet - y_0)$ have same value at (x^*, y^*), we obtain by assessing the latter inequality at this point,

$$d\Delta(V(x^*, y^*) - \varepsilon^*\varphi(x^* - x_0, y^* - y_0)) < 0. \qquad \textbf{(II.8)}$$

Thereby (II.7) and (II.8) make it absurd and so prove the claim. $\square$ (claim)

The claim is going to allow us to achieve the proof of the lemma. Let $r > 0$,

- if $r > R$ then thanks to the claim and because $\varphi(0,0) = 1$, we get for all $(x,y) \in \mathbb{R} \times [r; +\infty)$, $V(x,y) \geq \varepsilon_0 > 0$;

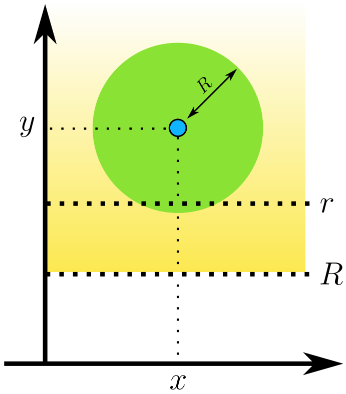


Figure F25 – *Illustration of the most simple case $r > R$. Because the claim is valid on the whole yellow half plane, one can put (x,y) anywhere we want above the line $y = r$.*

- if $0 < r \leq R$, then from one part, using the same argument as bellow, for all $(x,y) \in \mathbb{R} \times (R; +\infty)$, $V(x,y) \geq \varepsilon_0 > 0$. From another part, thanks again to the claim, for all $(x,y) \in \mathbb{R} \times (r; R]$, $V(x,y) \geq \varepsilon_0 \varphi\left(0, \frac{r}{2} - R\right) > 0$.

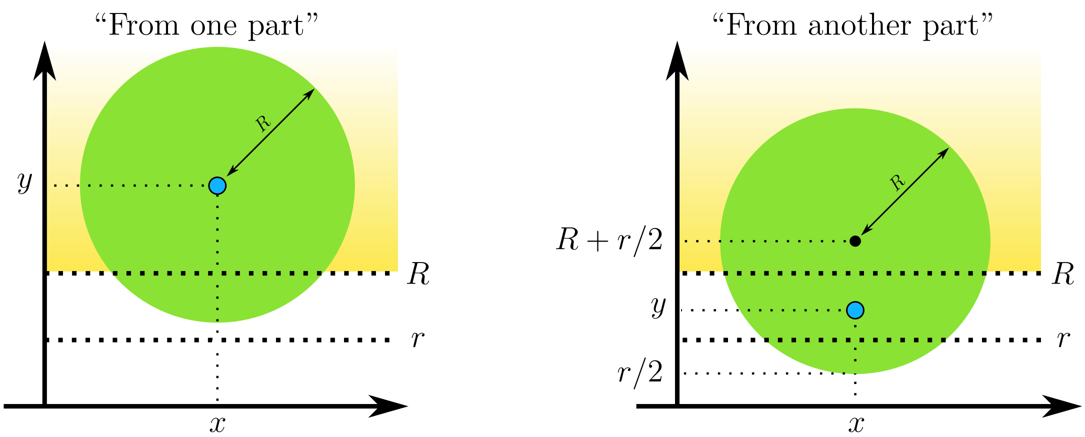


Figure F26 – *Illustration of the second case $0 < r \leq R$. The left part uses the same arguments as the first case. The right part uses the fact that φ is positive, radial and decays with respect to the radius of its argument.*

To conclude we have shown that

$$\inf_{\mathbb{R}\times[r;+\infty)} V \geq \varepsilon_0 > 0 \qquad \text{if } r > R,$$

and

$$\inf_{\mathbb{R}\times[r;+\infty)} V \geq \varepsilon_0 \varphi\left(0, \frac{r}{2} - R\right) > 0 \qquad \text{otherwise.}$$

So the result is established. □

Proof (Proposition 30) (Stationary solutions)
Let (U, V) be a bounded and non-negative solution of problem (II.5). We are seeking to show that $(U, V) \equiv (0, 0)$ or $(U, V) \equiv (1/\mu, 1)$. By looking at third line of (II.5), one sees that it is sufficient to prove that $V \equiv 0$ or $V \equiv 1$. That sufficient condition is again not surprising because of the equilibrium ratio between the mass of the population on the Field's frontier and on the Road. We reason according to whether V is positive or not.

- If V is not positive, then because V is assumed non-negative there exists some $(x_0, y_0) \in \mathcal{F}$ such that $V(x_0, y_0) = 0$. Then by applying the strong elliptic maximum principle (see page 119) to $\mathcal{L} := -d\Delta$, as $-V$ attains an interior maximum at (x_0, y_0) and verifies $\mathcal{L}(-V) = -f(V) \leq 0$, we get $V \equiv 0$ on $\mathcal{F}$. In that way one recovers the first solution we were aiming for.

- If V is positive, we therefore have to show that $V \equiv 1$ to achieve the proof.

Let then us suppose $V > 0$, we proceed in two steps:

$\boxed{\mathbf{1}}$ Show that $V \geq 1$.

To do this, assume by contradiction that $m := \inf\limits_{\mathcal{F}} V < 1$. Note because V is assumed positive, one necessary has $m \geq 0$. Take a sequence of points $(x_n, y_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ such that

$$\lim_{n\to\infty} V(x_n, y_n) = m.$$

There are two possible cases depending on whether m is obtained in the interior of the Field or on its frontier:

$\boxed{\text{First case}}$ (y_n) tends to 0. We set for $n \in \mathbb{N}$,

$$U_n := U(x + x_n) \qquad \text{and} \qquad V_n := V(x + x_n, y).$$

Thanks to standard elliptic estimates, both sequences (U_n) and (V_n) converge respectively (up to sub-sequences and locally uniformly) toward some functions $\widetilde{U}$ and $\widetilde{V}$ such that $\left(\widetilde{U}, \widetilde{V}\right)$ satisfies problem (II.5). We have then according to second line of that problem,

$$\Delta\widetilde{V} = -f\left(\widetilde{V}\right)/d. \qquad \textbf{(II.9)}$$

Furthermore,

$$\begin{aligned}
\widetilde{V}(0,0) &= \lim_{n\to\infty} V_n(0,0) \\
&= \lim_{n\to\infty} V(0 + x_n, 0) \\
&= V\left(\lim_{n\to\infty} x_n, 0\right) \\
&= V\left(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n\right) \\
&= \lim_{n\to\infty} V(x_n, y_n) \\
&= m,
\end{aligned}$$

and as $\widetilde{V}$ is continuous, there exists some vicinity of $(0,0)$ in $\mathbb{R}_+^2$ rated $\mathcal{V}$ in which $0 \leq \widetilde{V} \simeq m < 1$. Thus, because f is non-negative between 0 and 1, one obtains that $f\left(\widetilde{V}\right)$ is non-negative in $\mathcal{V}$ and so with (II.9), $\Delta\widetilde{V} \leq 0$. One can now apply the elliptic Hopf's lemma (see page 119) with $\mathcal{L} := -\Delta$: as $-\widetilde{V}$ attains a maximum at $(0,0)$ that is on the frontier of $\mathcal{V}$ and $\mathcal{L}\left(-\widetilde{V}\right) \leq 0$ in $\mathcal{V}$, we get

- either $\widetilde{V} \equiv m$ in $\mathcal{V}$,
- or $-\partial_y \widetilde{V}(0,0) < 0$.

First situation can be discarded, indeed, whether it would be true, we would have, by using lemma 31 (with r small enough), that $m = \inf\limits_{\mathbb{R}\times[r;+\infty)} \widetilde{V} > 0$. Whence, we be would getting the following contradiction

$$0 = \Delta\widetilde{V} \overset{(\text{II.9})}{=} -\frac{f(m)}{d} < 0 \ \ (\text{because } 0 < m < 1).$$

So we should be in the second situation given by Hopf's lemma that is $-\partial_y \widetilde{V}(0,0) < 0$. Using that in third line of problem (II.5), one reaches

$$\mu\widetilde{U}(x) - \widetilde{V}(x,0) < 0.$$

Assess that for $x = 0$,

$$\mu\widetilde{U}(0) - m < 0.$$

Hence

$$\inf_{\mathbb{R}} U = \inf_{\mathbb{R}} \widetilde{U} < \widetilde{U}(0) < \frac{m}{\mu}. \qquad \textbf{(II.10)}$$

We set now a minimizing sequence $(\widehat{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ that is $\lim\limits_{n\to\infty} U(\widehat{x}_n) = \inf\limits_{\mathbb{R}} U$ and take for $n \in \mathbb{N}$, $\widehat{U}_n(x) := U(x + \widehat{x}_n)$. Another time, the standard elliptic estimates assert that $\left(\widehat{U}_n\right)$ converges (up to sub-sequence and locally uniformly) toward a function $\widehat{U}$ such that $\widehat{U}(0) = \inf\limits_{\mathbb{R}} \widehat{U} = \inf\limits_{\mathbb{R}} U$, and $\left(\widehat{U}, V\right)$ satisfies system (II.5). So by using first line of that system,

$$-D\partial_{xx}\widehat{U} = V(x,0) - \mu\widehat{U},$$

and by assessing that in $x = 0$,

$$\begin{aligned} -D\partial_{xx}\widehat{U}(0) &= V(0,0) - \mu\widehat{U}(0) \\ &= m - \mu \inf_{\mathbb{R}} U \\ &\overset{(\text{II.10})}{>} 0. \end{aligned}$$

Furthermore, by minimality of $\widehat{U}$ in 0, we obtain $-D\partial_{xx}\widehat{U}(0) \leq 0$ so that's absurd.

$\boxed{\text{Second case}}$ (y_n) does not tends to 0.

Otherwise said, there exists a sub-sequence $(y_{n_k})_{k\in\mathbb{N}}$ and $r>0$ such that $y_{n_k}>r>0$ for all $k\in\mathbb{N}$. Thanks to lemma 31, and as $(x_{n_k},y_{n_k})\in\mathbb{R}\times[r;+\infty)$, so it is easy to see that

$$V\left(x_{n_k},y_{n_k}\right)\geq\inf_{\mathbb{R}\times[r;+\infty)}V>0.$$

Whence by taking the limit as k tends to $+\infty$, we get that $m>0$. We now pose for $k\in\mathbb{N}$,

$$\overline{V}_{n_k}(x,y):=V\left(x+x_{n_k},y+y_{n_k}\right).$$

Using the standard elliptic estimates, we obtain that $\left(\overline{V}_{n_k}\right)$ converges (up to sub-sequence and locally uniformly) toward a function $\breve{V}$ such that $\breve{V}(0,0)=m$ and $\left(U,\breve{V}\right)$ satisfies system (II.5). The function $\breve{V}$ is continuous so in a small-enough centred ball B_ρ (with $0<\rho<r$), we have

$$0<\breve{V}\simeq\breve{V}(0,0)=m<1.$$

If we write second line of system (II.5), we have

$$-d\Delta\breve{V}=f\left(\breve{V}\right)$$

and because m is strictly between 0 and 1 in B_ρ, $-d\Delta\breve{V}>0$. Yet $\breve{V}$ attains an interior minimum at $(0,0)$ so we also have in this point $-d\Delta\breve{V}\leq 0$; that's absurd.

Thereby in both cases, $V\geq 1$. $\square$ (1)

$\boxed{\mathbf{2}}$ Show that $V\leq 1$.

That part of the proof follows the same arguments as in $\boxed{1}$; we thus take the same notations and give less details. Assume by contradiction that $M:=\sup\limits_{\mathcal{F}}V>1$. One takes a sequence $(x_n,y_n)_{n\in\mathbb{N}}\subset\mathcal{F}$ such that

$$\lim_{n\to\infty}V\left(x_n,y_n\right)=M.$$

Again, there are two possible cases depending on whether M is obtained in the interior of the Field or on its frontier.

$\boxed{\text{First case}}$ (y_n) tends to 0. We set for $n\in\mathbb{N}$,

$$U_n:=U\left(x+x_n\right)\qquad\text{and}\qquad V_n:=V\left(x+x_n,y\right).$$

Thanks to standard elliptic estimates, both sequences (U_n) and (V_n) converge respectively (up to sub-sequences and locally uniformly) toward some functions $\widetilde{U}$ and $\widetilde{V}$ such that $\left(\widetilde{U},\widetilde{V}\right)$ satisfies problem (II.5). We furthermore have

$$\Delta\widetilde{V}=-f\left(\widetilde{V}\right)/d,\qquad\textbf{(II.11)}$$

and $\widetilde{V}(0,0) = M$. Hopf's lemma give us, in a same vein as (II.10)

$$\sup_{\mathbb{R}} U = \sup_{\mathbb{R}} \widetilde{U} > \widetilde{U}(0) > \frac{M}{\mu}. \qquad \textbf{(II.12)}$$

We set now a maximizing sequence $(\widehat{x}_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ that is $\lim\limits_{n\to\infty} U(\widehat{x}_n) = \sup\limits_{\mathbb{R}} U$ and take for $n \in \mathbb{N}$, $\widehat{U}_n(x) := U(x + \widehat{x}_n)$. Another, time the standard elliptic estimates assert that $\left(\widehat{U}_n\right)$ converges (up to sub-sequence and locally uniformly) toward a function $\widehat{U}$ such that $\widehat{U}(0) = \sup\limits_{\mathbb{R}} \widehat{U} = \sup\limits_{\mathbb{R}} U$, and $\left(\widehat{U}, V\right)$ satisfies system (II.5). So by using first line of that system,

$$-D\partial_{xx}\widehat{U} = V(x,0) - \mu\widehat{U},$$

and by assessing that in $x = 0$,

$$\begin{aligned}
-D\partial_{xx}\widehat{U}(0) &= V(0,0) - \mu\widehat{U}(0) \\
&= M - \mu \sup_{\mathbb{R}} U \\
&\overset{(\text{II.12})}{<} 0.
\end{aligned}$$

Furthermore, by maximality of $\widehat{U}$ in 0, we obtain $-D\partial_{xx}\widehat{U}(0) \geq 0$ so that's absurd.

$\boxed{\text{Second case}}$ (y_n) does not tends to 0.

Otherwise said, there exists a sub-sequence $(y_{n_k})_{k\in\mathbb{N}}$ and $r > 0$ such that $y_{n_k} > r > 0$ for all $k \in \mathbb{N}$. We now pose for $k \in \mathbb{N}$,

$$\overline{V}_{n_k}(x,y) := V(x + x_{n_k}, y + y_{n_k}).$$

Using the standard elliptic estimates, we obtain that $\left(\overline{V}_{n_k}\right)$ converges (up to sub-sequence and locally uniformly) toward a function $\breve{V}$ such that $\breve{V}(0,0) = M$ and $\left(U, \breve{V}\right)$ satisfies system (II.5). The function $\breve{V}$ is continuous so in a small-enough centred ball B_ρ (with $0 < \rho < r$), we have

$$0 < \breve{V} \simeq \breve{V}(0,0) = M > 1.$$

If we write second line of system (II.5), we have

$$-d\Delta\breve{V} = f\left(\breve{V}\right)$$

and because M is larger than 1 in B_ρ, $-d\Delta\breve{V} < 0$. Yet $\breve{V}$ attains an interior maximum at $(0,0)$ so we also have in this point $-d\Delta\breve{V} \geq 0$; that's absurd.

Thereby in both cases, $V \leq 1$. $\square_{(2)}$

Consequently, $V \equiv 1$. $\square$

One comes back to the initial temporal problem (II.2), proposition 30 actually allows us to show the following invasion result:

Theorem 32 (Hair Trigger Effect on the Field-Road)
For all bounded initial datum $(u_0, v_0) \geq (0,0)$ but $\not\equiv (0,0)$, the solution (u,v) of the Field-Road Fisher-KPP Cauchy problem (II.2)–(II.3) satisfies

$$\lim_{t\to\infty} \left(u\left(t,x\right), v\left(t,x,y\right)\right) = \left(\frac{1}{\mu}, 1\right)$$

locally uniformly in the whole half plane $\mathbb{R}^2_+$.

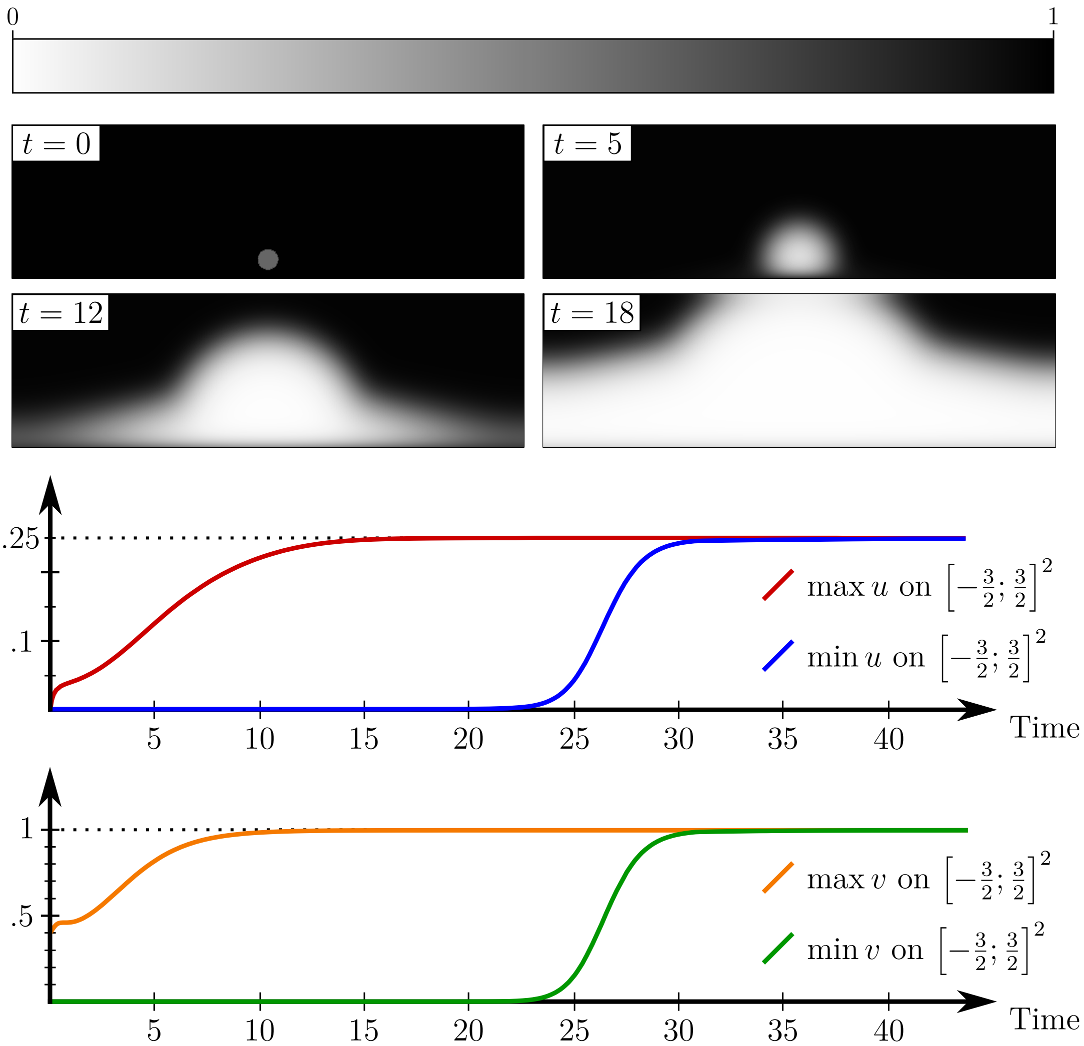


Figure F27 – *Simulation of the Hair Trigger Effect on the Field-Road using the logistic reaction $f(u) = u(1-u)$, $D = 1$, $d = 0.01$, $\mu = 4$ and starting from the compactly supported initial datum $(u_0, v_0) := \left(\frac{2}{5}\mathbb{1}_{\mathcal{B}_{\frac{1}{5}}\left(0,\frac{3}{8}\right)}, 0\right)$. On the top of the figure, one sees four snapshots of the Field solution v through the window $\left[-\frac{3}{2};\frac{3}{2}\right]^2$, and on* *the bottom the local* min *and* max *of the solution* (u,v) *have been represented in that same window. Observe how population density tends toward* $(1/\mu,1)$ *locally and uniformly in space.*

Proof (32) (Hair Trigger Effect on the Field-Road)
The main idea of this proof is to create two solutions $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ of (II.2), the first increasing in time at fixed place and the second decreasing in time at fixed place; we show then these functions tends toward $(1/\mu,1)$ locally and uniformly in space as t tends to ∞; and finally, one puts our solution (u,v) between these two bounds to get the result.

Starting as for the proof of lemma 31 (page 48), we denote by (λ,φ) the couple principal-(eigenvalue/eigenfunction) of $-d\Delta$ with Dirichlet boundary condition in $\mathcal{B}_R$ as defined in section 2 of the Toolbox part page 118. Take $R>0$ large enough so that

$$\lambda < \frac{f'(0)}{2d},$$

one poses for $\varepsilon>0$,

$$\underline{V} := \varepsilon\varphi\left(x, y-R-1\right).$$

It has been shown in proof of lemma 31 that for ε small enough, $-d\Delta\underline{V} < f\left(\underline{V}\right)$ in $\mathcal{B}_R\left(0,R+1\right)$. Furthermore, by taking $\varepsilon\leq 1$, we have $\underline{V}\leq 1$. Extend $\underline{V}$ by zero outside $\mathcal{B}_R\left(0,R+1\right)$ in the Field and recall that extension $\underline{V}$. One can prove that $(0,\underline{V})$ is a generalised sub-solution of (II.2) in the meaning of proposition 29 (page 47). Let $(\underline{u},\underline{v})$ be the solution of (II.2) starting from initial condition $(0,\underline{V})$. In the same vein as corollary 19 (page 23) we may show that, because $(\underline{u},\underline{v})$ is sub-solution of its own system at time $t=0$, both $\underline{u}$ and $\underline{v}$ are increasing with respect to the time at fixed location in space.

Now we have created an increasing solution, we are looking for a decreasing one. We pose

$$\overline{U} := \max\left(\sup_{\mathbb{R}}\left(u_0\right); \frac{1}{\mu}\sup_{\mathcal{F}}\left(v_0\right); \frac{1}{\mu}\right),$$

$$\overline{V} := \max\left(\mu\sup_{\mathbb{R}}\left(u_0\right); \sup_{\mathcal{F}}\left(v_0\right); 1\right).$$

Although $\overline{U}$ and $\overline{V}$ do not share de same domain, note that "$\overline{V}=\mu\overline{U}$" ; otherwise said,

$$\left[\overline{U} = \sup_{\mathbb{R}}\left(u_0\right)\right] \iff \left[\overline{V} = \mu\sup_{\mathbb{R}}\left(u_0\right)\right]$$

$$\left[\overline{U} = \frac{1}{\mu}\sup_{\mathcal{F}}\left(v_0\right)\right] \iff \left[\overline{V} = \sup_{\mathcal{F}}\left(v_0\right)\right]$$

$$\left[\overline{U} = \frac{1}{\mu}\right] \iff \left[\overline{V} = 1\right].$$

It is then easy, thanks to its constant nature, to show that $\left(\overline{U},\overline{V}\right)$ is super-solution of (II.2). Let $(\overline{u},\overline{v})$ be the solution of (II.2) starting from the initial state $\left(\overline{U},\overline{V}\right)$. By the same argument we use for $(\underline{u},\underline{u})$, both $\overline{u}$ and $\overline{v}$ are decreasing with respect to the time at fixed location in space.

Hence one disposes of

- $(\underline{u},\underline{v})$ which is bounded from above and increasing at fixed location with respect to the time, and so converges point-wise up to $(\underline{u}_\infty,\underline{v}_\infty)$ as t tends to $+\infty$,
- $(\overline{u},\overline{v})$ which is bounded from bellow and decreasing at fixed location with respect to the time, and so converges point-wise down to $(\overline{u}_\infty,\overline{v}_\infty)$ as t tends to $+\infty$.

Then parabolic estimates assert us that both of these two convergences are more than point-wise, namely locally uniformly in space. So one can "take the limit" in system (II.2) which leads us to the stationary problem (II.5) and therefore, one actually has

$$(\underline{u}_\infty,\underline{v}_\infty) = (\overline{u}_\infty,\overline{v}_\infty) \equiv (1/\mu, 1)\,.$$

The last thing we have to do to complete this proof is slipping our solution (u,v) between $(\underline{u},\underline{v})$ and $(\overline{u},\overline{v})$ for t large enough. Getting (u,v) bellow $(\overline{u},\overline{v})$ is easy thanks to the comparison principle because it is the case at $t=0$. Therefore,

$$\limsup_{t\to\infty}\left(u\left(t,x\right),v\left(t,x,y\right)\right) \leq \left(\frac{1}{\mu},1\right) \text{ locally uniformly in space.}$$

We cannot do the same for $(\underline{u},\underline{v})$ because $\underline{v}$ is positive and v_0 may not. (Note $\underline{u}$ is not a problem because it starts from $0 \leq u_0$.) Since we know (thanks to the comparison principle) that v is positive for all positive time, we wait the instant $t=1$ so that v has time to peel off. By taking (if necessary) ε smaller in $\underline{V}$ definition, it skips under $v\left(\bullet+1,\bullet,\bullet\right)$, so the comparison principle may be applied and yields

$$\liminf_{t\to\infty}\left(u\left(t+1,x\right),v\left(t+1,x,y\right)\right) \geq \left(\frac{1}{\mu},1\right) \text{ locally uniformly in space.}$$

Thereby we recover

$$\lim_{t\to\infty}\left(u\left(t,x\right),v\left(t,x,y\right)\right) = \left(\frac{1}{\mu},1\right) \text{ locally uniformly in space}$$

that's what we were aiming for. □

II.6 Exponential super-solutions

We have shown in the previous section that there is Hair Trigger Effect for the Field-Road model with Fisher-KPP reaction, that is any non-degenerate given population, no matter how small, invade the whole half plane $\mathbb{R}^2_+$ locally uniformly in space. We wonder now about the speed of space invasion. In the current section we shall build the tools which will allow us to control the speed of space invasion from above. More precisely we will create an exponential super-solution above the solution and which will cross the space at constant speed. That speed shall be an upper bound for the speed of the solution so it is in our interest to make the exponential super-solution travel as slowly as possible to get the best control.

Before starting, we announce a result on the speed of propagation solutions already established in by Aronson and Weinberger in [2] which concern the classical R-D system in the whole space $\mathbb{R}^2$. Of course, we expect that a Road diffusion D significantly larger than the diffusion d in the Field would enhance the invasion speed of the classical system.

Theorem 33 (Aronson-Weinberger) (Asymptotic spreading speed) (1978) Let define $C_{\text{KPP}} := 2\sqrt{df'(0)}$ and consider the R-D Cauchy problem in $\mathbb{R}^2$,

$$\begin{cases} \partial_t u = d\Delta u + f(u) & (t,X) \in (0;\infty) \times \mathbb{R}^N \\ u(0,X) = u_0(X) & X \in \mathbb{R}^N. \end{cases} \tag{II.13}$$

Imagine a viewer at any initial position $Y \in \mathbb{R}^N$ and straightly moving in some direction at speed $c > 0$. That theorem presents two results.

$\boxed{\textbf{The viewer is too slow}}$ that is whether $c \in (0; C_{\text{KPP}})$, then the viewer suffers the invasion and only sees $u = 1$ “behind him” ([a]) as t become large. Otherwise said,

$$\lim_{t\to\infty}\left(\inf_{|X-Y|\leq ct} u(t,X)\right) = 1.$$

$\boxed{\textbf{The viewer is too fast}}$ that is whether $c \in (C_{\text{KPP}};\infty)$, then the viewer outrun the invasion and only sees $u = 0$ ”in front of” ([b]) him as t become large. Otherwise said,

$$\lim_{t\to\infty}\left(\sup_{|X-Y|\geq ct} u(t,X)\right) = 0.$$

[a] That is his vision sight is in the time-expanding ball starting from his initial position and whom radius equals ct.

[b] That is his vision sight is the complement of the set mentioned in the previous note.

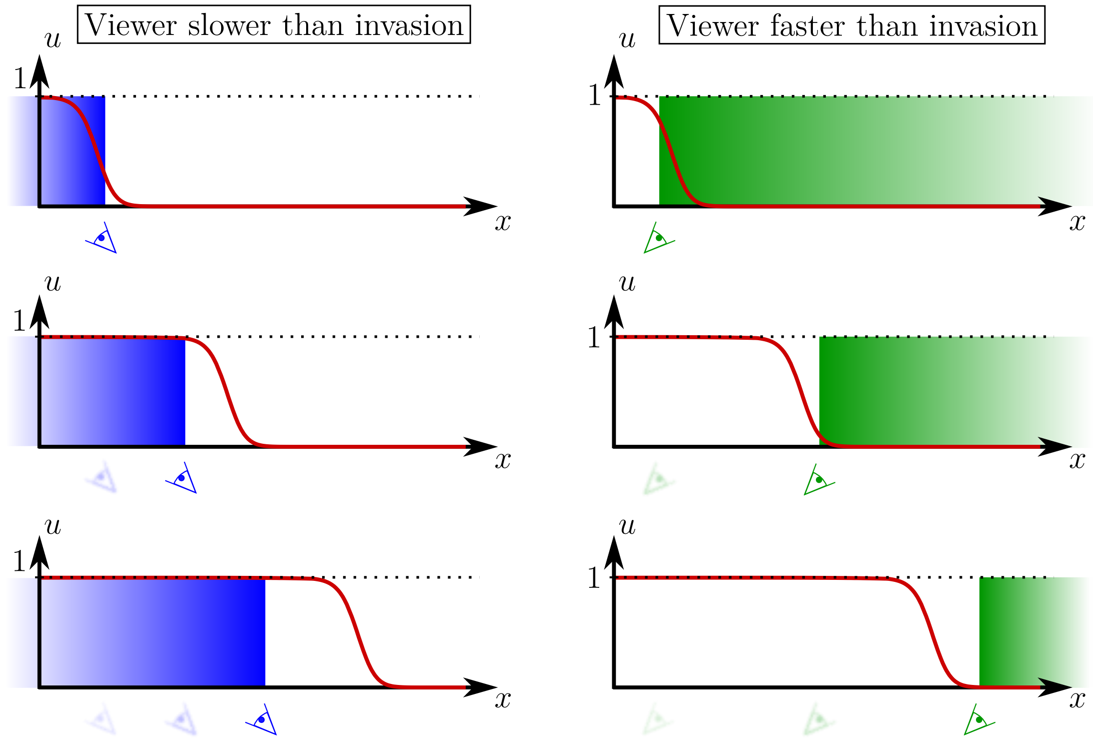


Figure F28 – *Illustration of two viewers represented at three different times. The viewer moves slower than C_{KPP} which is the invasion speed of the solution whereas the viewer moves faster than C_{KPP}. Observe that in large time, the slow-viewer only sees $u = 1$ "behind him" whereas the fast-viewer only sees $u = 0$ "in front of him".*

We can now look for the exponential super-solutions. Recall that, with the KPP-hypothesis (see page 8), $f(v) \leq vf'(0)$ for all positive v. Therefore, all solutions of system

$$\begin{cases} \partial_t u - D\partial_{xx} u = v(t,x,0) - \mu u & (t,x,0) \in (0;\infty) \times \mathcal{R} \\ \partial_t v - d\Delta v = vf'(0) & (t,x,y) \in (0;\infty) \times \mathcal{F} \\ -d\partial_y v(t,x,0) = \mu u(t,x) - v(t,x,0) & (t,x,0) \in (0;\infty) \times \mathcal{R}. \end{cases} \quad \textbf{(II.14)}$$

should be (provided they exist) super-solutions of system (II.2). One defines then for

$$\alpha > 0, \qquad \beta \in \mathbb{R}, \qquad \gamma > 0, \qquad c \in \mathbb{R},$$

the function

$$\begin{pmatrix} \overline{u}(t,x) \\ \overline{v}(t,x,y) \end{pmatrix} := \begin{pmatrix} e^{\alpha(x+ct)} \\ \gamma e^{\alpha(x+ct)-\beta y} \end{pmatrix}$$

which will be the exponential super-solution we mentioned at the beginning of the section. Let us at first justify that it is a well tool to deal with some propagation speed. It is easy to see that, given $\ell > 0$, the level-set $\{\overline{u} = \ell\}$ is reduced to one point which travels the $\mathbb{R}$ line at constant speed c, on the left if $c > 0$ and on the right if $c < 0$ as one sees on figure (*F29*)

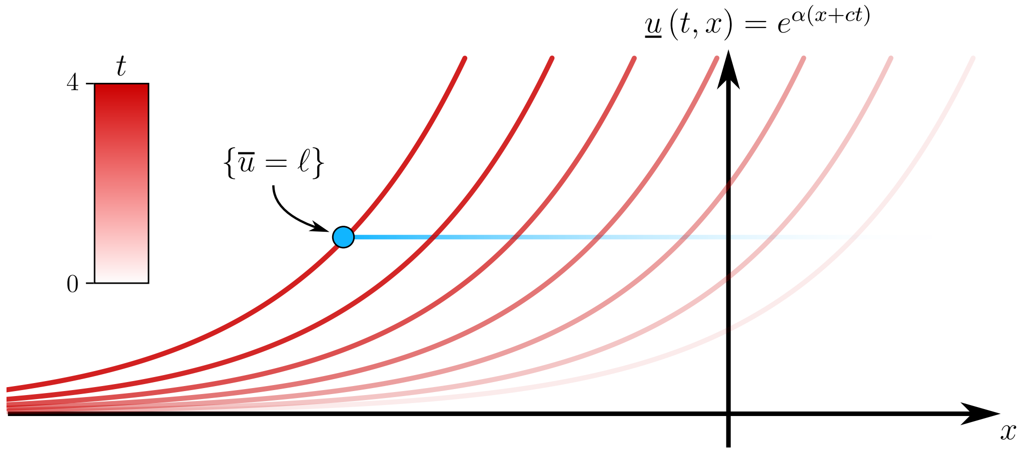


Figure F29 – *Illustration of the u-part of the travelling exponential super-solution with the speed $c = 1$ and $\alpha = 1$.*

We actually have the same thing for $\overline{v}$; indeed, take the level-set $\{\overline{u} = \ell\}$ with the innocent choice $\ell = \gamma$. There comes:

$$\begin{aligned}
[(x,y) \in \{\overline{u} = \gamma\}] &\iff [v(t,x,y) = \gamma] \\
&\iff \left[\gamma e^{\alpha(x+ct)-\beta y} = \gamma\right] \\
&\iff [\alpha x - \beta y + \alpha c t = 0] \\
&\iff [(x,y) \in \Delta_\gamma]
\end{aligned}$$

where Δ_γ is the half line in $\mathbb{R}^2_+$ starting from the point $(-ct, 0)$ and directed by the vector $\begin{pmatrix}\beta/\alpha \\ 1\end{pmatrix}$. In a more general case, the level-set $\{\overline{u} = \ell\}$ is the half line

$$\Delta_\ell := \underbrace{\begin{pmatrix}\ln\left(\frac{\ell}{\gamma}\right)/\alpha - ct \\ 0\end{pmatrix}}_{\text{call that point } A} + s\begin{pmatrix}\beta/\alpha \\ 1\end{pmatrix}, \qquad \text{with } s \in \mathbb{R}_+.$$

Note the direction of that line does not depend on time and therefore the latter cross the half space following $\begin{pmatrix}-1 \\ 0\end{pmatrix}$ at speed c. Another time, the situation has been illustrated on figure (*F30*)

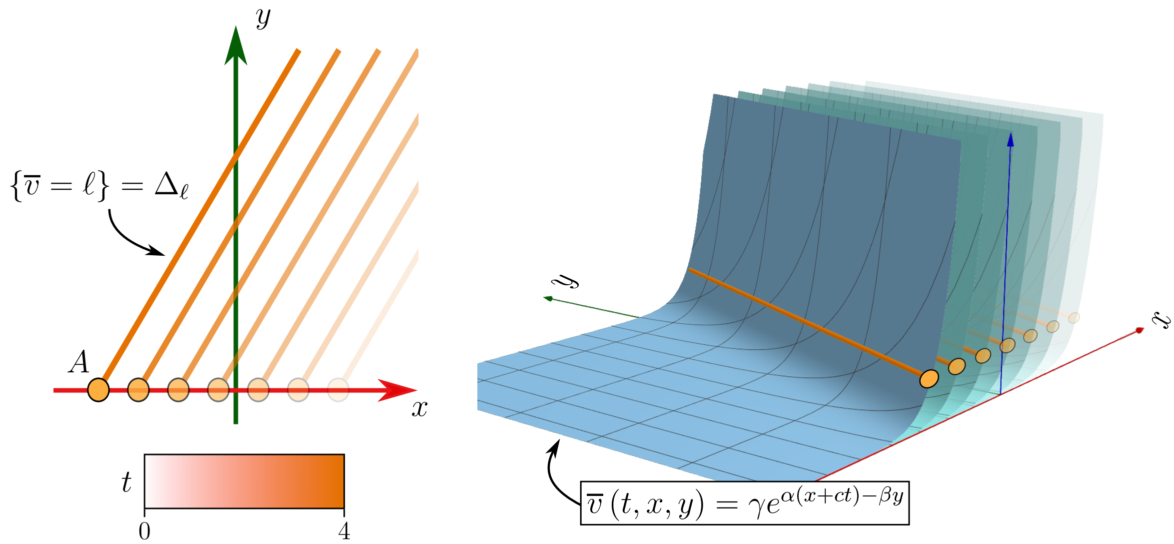


Figure F30 – *Illustration of the v-part of the travelling exponential super-solution with the speed $c = 1$ and $\alpha, \beta, \gamma = 1$.*

Our aim is now, for given c, to find the "good" parameters α, β, γ for which $(\overline{u}, \overline{v})$ is a solution of problem (II.14). Note that the diffusion coefficients d and D are fixed. If we insert $(\overline{u}, \overline{v})$ inside (II.14), we are lead to the three following relations respectively coming from each of the three equations of the system.

$$\begin{cases} -D\alpha^2 + c\alpha = \gamma - \mu & \textbf{(i)} \\ -d\alpha^2 + c\alpha = f'(0) + d\beta^2 & \textbf{(ii)} \\ d\beta\gamma = \mu - \gamma. & \textbf{(iii)} \end{cases} \qquad \textbf{(II.15)}$$

$\boxed{\textbf{(i) and (iii)}}$ Remark first that by modifying (**iii**), one gets

$$\gamma = \frac{\mu}{1 + d\beta}$$

and so, because β induces the value of γ, we just have to find some conditions on α and β. Solving now the quadratic equation (**i**) whom α is the parameter, we find

$$\alpha_D^{\pm}(c, \beta) := \frac{1}{2D}\left(c \pm \sqrt{c^2 + \frac{4\mu dD\beta}{1 + d\beta}}\right).$$

To get the inside of the square root non-negative, β has to be chosen lower than $-1/d$ or greater than $-\frac{c^2}{d(c^2+4\mu D)}$. Note that first case has to be discarded because γ is chosen positive. One sets now

Graph of α_D^+

$$\Gamma_{c,D} := \Gamma_{c,D}^{+} \cup \Gamma_{c,D}^{-}$$

Graph of α_D^-

And we get so [(**i**) and (**iii**)] $\iff$ [$(\beta, \alpha) \in \Gamma_{c,D}$]. One refers the reader to figure (*F32*) for the shape of $\Gamma_{c,D}$.

$\boxed{\textbf{(ii)}}$ Let's now see equation (**ii**) which is equivalent to

$$\left(\alpha - \frac{c}{2d}\right)^2 + \beta^2 = \frac{c^2 - C_{\mathrm{KPP}}^2}{4d^2}.$$

From this viewpoint, (β, α) has to be on the circle centred on $(0, c/2d)$ and of radius

$$\beta_{\mathrm{KPP}}(c) := \frac{\sqrt{c^2 - C_{\mathrm{KPP}}^2}}{2d}.$$

Note we require that $c \geq C_{\mathrm{KPP}}$ otherwise such a circle cannot exist and so **(ii)** cannot be verified. (Note $\Gamma_{C_{\mathrm{KPP}},d}$ is a single point.) One represents that circle with two functions that respectively correspond to the upper and lower piece of it:

$$\alpha_d^{\pm}(c, \beta) := \frac{c \pm \sqrt{c^2 - C_{\mathrm{KPP}}^2 - 4d^2\beta^2}}{2d}.$$

Like as bellow, one takes

$$\Gamma_{c,d} := \overbrace{\Gamma_{c,d}^{+}}^{\text{Graph of } \alpha_d^+} \cup \underbrace{\Gamma_{c,d}^{-}}_{\text{Graph of } \alpha_d^-}$$

And we get so [(**ii**)] $\iff$ [$(\beta, \alpha) \in \Gamma_{c,d}$]. One refers again the reader to figure (*F32*) for the shape of $\Gamma_{c,d}$.

Hence the couples (α, β) we are looking for are at the intersection $\Gamma_{c,D} \cap \Gamma_{c,d}$ and we thus have to find for which $c \geq C_{\mathrm{KPP}}$ that intersection is non-empty. Recall that our aim is having c as small as possible. We distinguish three cases depending on the position of D relatively to $2d$, or in an equivalent way, on the position of compared to (see figure (*F31*)).

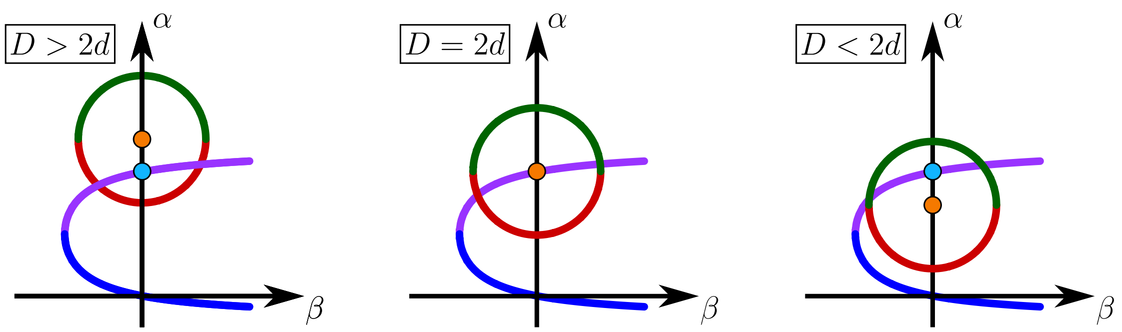


Figure F31 – *Configuration of the graph of compatibility whether the position of D relatively to $2d$. Note that on the draw, for $D > 2d$ and $D < 2d$, $\Gamma_{c,D}$ and $\Gamma_{c,d}$ intersect but it is not always the case. As you may see on figure (F32).*

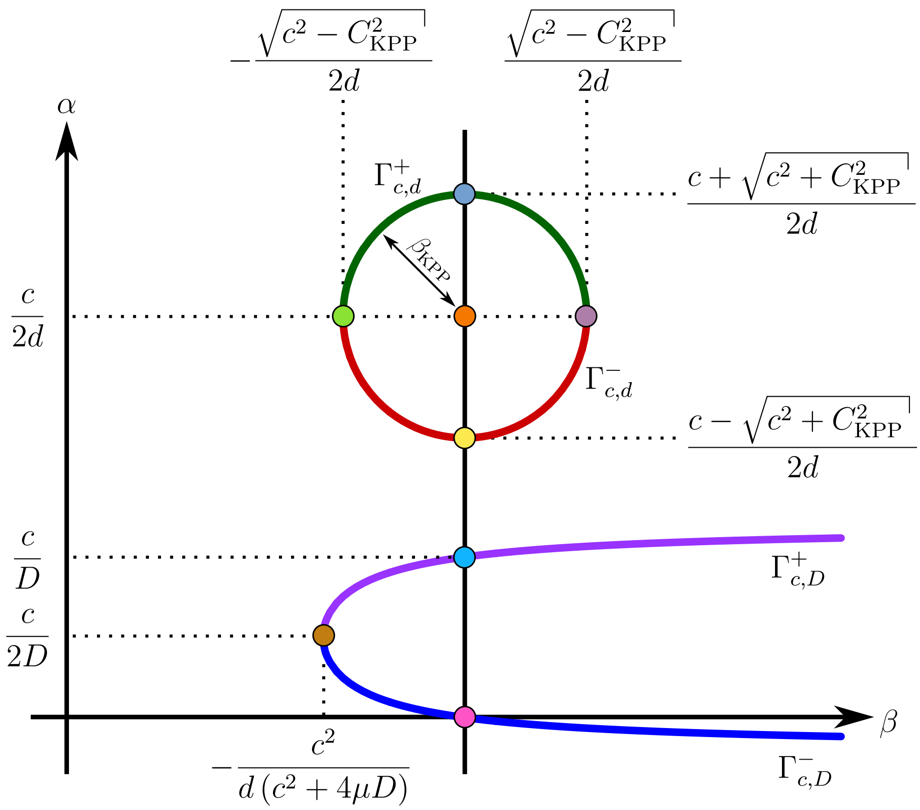


Figure F32 – *Compatibility of parameters α and β: $(\overline{u}, \overline{v})$ is solution of (II.14) when $\Gamma_{c,D}$ and $\Gamma_{c,d}$ intersect.*

$\boxed{\text{Case } D > 2d}$ In that situation, we have $c/D < c/2d$, that is 🟠 is strictly above 🔵. Therefore, there is no intersection between $\Gamma_{c,d}$ and $\Gamma_{c,D}$ when $c = C_{\text{KPP}}$ because in that case $\Gamma_{c,d}$ is reduced to the singleton $\{(0, c/2d)\} = \{$🟠$\}$. Observe now that $\partial_c \alpha_d^- < 0$ and that

$$\lim_{c \to \infty} \alpha_d^- (c, 0) = \lim_{c \to \infty} \frac{c - \sqrt{c^2 + c_{KPP}^2}}{2d} = 0.$$

Therefore, the point 🟡 vertically get down to the origin. Furthermore, we can also see that $\partial_c \alpha_D^+ > 0$ and that

$$\lim_{c \to \infty} \alpha_D^+ (c, 0) = \lim_{c \to \infty} \frac{c}{D} = +\infty.$$

Hence the point 🔵 vertically get up toward infinity. Thereby $\Gamma_{c,D}^+$ and $\Gamma_{c,d}^-$ have to cross for some $c > C_{\text{KPP}}$. Because $\alpha_{c,d}^- - \alpha_{c,D}^+$ is strictly convex, there exists a unique $C_* = C_* (\mu, d, D) > C_{\text{KPP}}$ such that

- $\Gamma_{C_*,d}^-$ and $\Gamma_{C_*,D}^+$ are tangent and thus $\Gamma_{C_*,d}$ and $\Gamma_{C_*,D}$ intersect exactly once,
- for all $c \in [C_{\text{KPP}}; C_*)$, $\Gamma_{c,d}$ and $\Gamma_{c,D}$ do not intersect.

That amount C_* is the minimal speed we are looking for.

$\boxed{\text{Case } D = 2d}$ In that situation, we have $c/D = c/2d$, that is $\textcolor{orange}{\bullet}$ and $\textcolor{cyan}{\bullet}$ share the same place. Then for $c = C_{\text{KPP}}$, $\Gamma_{C_*,d} = \{\textcolor{orange}{\bullet}\}$ and $\Gamma_{C_*,D}$ intersect at $\textcolor{orange}{\bullet}$ and thus $C_* := C_{\text{KPP}}$ is the minimal speed so that equations of (II.15) are compatible.

$\boxed{\text{Case } D < 2d}$ In that situation, we have $c/D > c/2d$, that is $\textcolor{orange}{\bullet}$ is strictly bellow $\textcolor{cyan}{\bullet}$. Remind that for $\beta > -\frac{c^2}{d(c^2+4\mu D)}$ fixed, $\alpha^{\pm}_{c,d}(c,\beta)$ denote the two roots of the quadratic equation (**i**). Take now $c = C_{\text{KPP}}$ and $(\beta, \alpha) = (0, C_{\text{KPP}}/2d) = \textcolor{orange}{\bullet}$.

- We set $\gamma = \mu$ to get (**iii**).
- (**ii**) is clearly satisfied because $\Gamma_{C_{\text{KPP}},d} = \{(0, C_{\text{KPP}}/2d)\} = \{\textcolor{orange}{\bullet}\}$.
- Finally, because, from one hand $\textcolor{orange}{\bullet}$ is between $\textcolor{cyan}{\bullet}$ and $\textcolor{magenta}{\bullet}$, otherwise said, $C_{\text{KPP}}/2d$ is between the two roots of (**i**), and from another hand $-D < 0$, we get

$$-D\alpha^2 + c\alpha \geq \gamma - \mu.$$

That is the function $(\overline{u}, \overline{v})$ taken with $(\alpha, \beta, \gamma) := (C_{\text{KPP}}/2d, 0, \mu)$ is actually a super-solution of system (II.14) and then, by transitivity, a super-solution of (II.2). What's why one chooses $C_* := C_{\text{KPP}}$ without any further considerations on that case.

To conclude that section, we have therefore found some exponential super-solutions $(\overline{u}, \overline{v})$ of problem (II.2) travelling at speed

- $C_* > C_{\text{KPP}}$ if $D > 2d$,
- $C_* = C_{\text{KPP}}$ if $0 \leq D \leq 2d$.

II.7 Asymptotic spreading speed C_*

In the continuity of the previous section, we give here a result concerning the asymptotic spreading speed of solutions for the Field-Road model. As mentioned above it is expected that this asymptotic speed depends on how individuals move fast on the Road compared to the Field.

We consider in the sequence the amount $C_* = C_*(\mu, d, D)$ as defined in section II.6. Before giving the "Field-Road version" of theorem 13 (page 58) let us remind that

- $C_* = C_{\text{KPP}}$ if $D \leq 2d$,
- $C_* > C_{\text{KPP}}$ if $D > 2d$.

C_* is actually the asymptotic spreading speed of the solution. Remark there are, as predicted, two regimes of speed depending on the position of D relatively to $2d$:

- if $D \leq 2d$, then the model behaves as the classical one in $\mathbb{R}^N$,
- if $D > 2d$, then the population invades the space strictly faster than for the classical model in $\mathbb{R}^N$.

Theorem 34 (Berestycki *et al.*) (Asymptotic spreading speed (Field-Road))
Let (u, v) be the solution of the Fisher-KPP Field-Road problem (II.2) starting from the initial datum (u_0, v_0) which is supposed

- non-negative,
- non-zero,
- smaller than $\left(\frac{1}{\mu}, 1\right)$,
- compactly supported.

Then (there is HTE and) there is an asymptotic spreading speed equals to C_* following the x-direction; otherwise said:

Too slow viewer For all $0 < c < C_*$,

$$\lim_{t\to\infty}\left(\inf_{|x|\le ct}\left(u\left(t,x\right),v\left(t,x,y\right)\right)\right)=\left(\frac{1}{\mu},1\right).$$

Too fast viewer For all $c > C_*$,

$$\lim_{t\to\infty}\left(\sup_{|x|\ge ct}\left(u\left(t,x\right),v\left(t,x,y\right)\right)\right)=\left(0,0\right).$$

The proof of theorem 34 consists in finding some sub- and super-solutions which shall surround (u, v). The (exponential) super-solutions have already been devised in section II.6 so we are now interested in setting up some sub-solutions. Let us consider the following linearised system in the moving framework penalised by some $\delta > 0$:

$$\begin{cases} \partial_t u - D\partial_{xx}u + c\partial_x u = v\left(t,x,0\right) - \mu u & (t,x,0)\in(0;\infty)\times\mathcal{R} \\ \partial_t v - d\Delta v + c\partial_x v = v\left(f'\left(0\right)-\delta\right) & (t,x,y)\in(0;\infty)\times\mathcal{F} \\ -d\partial_y v\left(t,x,0\right) = \mu u\left(t,x\right) - v\left(t,x,0\right) & (t,x,0)\in(0;\infty)\times\mathcal{R}. \end{cases} \tag{II.16}$$

Lemma 35
Consider problem (II.16). We present here two analogous results depending on the regime imposed by the ratio D/d.

$\boxed{1}$ Assume $D > 2d$, then for some $c < C_*$ close enough to C_*, there exists $\delta > 0$ such that problem (II.16) admits a

- non-negative,
- non-zero,
- compactly supported

generalised sub-solution in the meaning of proposition 29.

$\boxed{2}$ Assume $0 \le D \le 2d$, then for $c \in (0; C_{\mathrm{KPP}}) = (0; C_*)$, there exists $\delta > 0$ such that problem (II.16) admits a

- non-negative,
- non-zero,
- compactly supported

generalised sub-solution in the meaning of proposition 29.

We shall admit part $\boxed{1}$ of lemma 35 and only show part $\boxed{2}$ whom the reasoning is a classical one.

Proof (Lemma 35) $\boxed{2}$
Let c be in $(0; C_{\mathrm{KPP}})$.

Claim. For $\delta \in (0; f'(0) - c^2/4d)$, the equation

$$-d\Delta V + c\partial_x V = \left(f'(0) - \frac{\delta}{2}\right) V \qquad (x,y) \in \mathcal{F} \qquad \textbf{(II.17)}$$

owns a compactly supported sub-solution.

For $x_0 \in \mathbb{R}_+^*$, consider the Dirichlet problem

$$\begin{cases} -d\Phi'' + c\Phi' = (f'(0) - \delta)\,\Phi & x \in (-x_0; x_0) \\ \Phi(\pm x_0) = 0. \end{cases} \qquad \textbf{(II.18)}$$

One assess the characteristic equation of first line of (II.18):

$$-dr^2 + cr + (\delta - f'(0)) = 0 \qquad \textbf{(II.19)}$$

whom discriminant $\Delta = c^2 + 4d(\delta - f'(0))$ is negative if and only if $\delta \in (0; f'(0) - c^2/4d)$. Take then δ in that set, (II.19) owns so two complex roots which are

$$r_0 = \frac{c}{2d} \pm i \underbrace{\frac{\sqrt{f'(0) - \delta - c^2/4d}}{\sqrt{d}}}_{\text{let us call that } \omega.}$$

Therefore there exists $A, B \in \mathbb{R}$ such that

$$\Phi(x) = e^{(c/2d)x}\left(A\cos(\omega x) + B\sin(\omega x)\right).$$

By choosing $x_0 := \frac{\pi}{2\omega}$, the Dirichlet boundary conditions provide $A = 1$ and $B = 0$; so

$$\Phi(x) = e^{(c/2d)x}\cos(\omega x).$$

Let now $\psi = \psi_R$ the principal eigenfunction (see Toolbox page 118) for $-\partial_{yy}$ in $(-R; R)$ provided with Dirichlet boundary conditions. Take R small enough so that the eigenvalue associated to ψ is smaller than $\delta/2d$. We assert then that the function V defined on $\mathcal{F}$ by

$$V(x,y) := \begin{cases} \Phi(x)\,\psi(y + R + 1) & \text{if } (x,y) \in \left(-\frac{\pi}{2\omega}; \frac{\pi}{2\omega}\right) \times (1; 2R+1) \\ 0 & \text{otherwise} \end{cases}$$

is the sub-solution of (II.17) announced by the claim. Indeed,

$$\begin{aligned}-d\Delta V + c\partial_x V - \left(f'(0) - \frac{\delta}{2}\right) V &= -d\left(\psi\Phi'' + \Phi\psi''\right) + c\Phi'\psi - \left(f'(0) - \frac{\delta}{2}\right)\Phi\psi \\ &\leq \psi\left(d\Phi'' + \frac{\delta}{2}\Phi + c\Phi' - \left(f'(0) - \frac{\delta}{2}\right)\Phi\right) \\ &= \psi \underbrace{\left(d\Phi'' + c\Phi' - \left(f'(0) - \delta\right)\Phi\right)}_{\text{remind that } \Phi \text{ is solution of (II.18)...}} \\ &= 0.\end{aligned}$$

Thereby the claim is established. $\square$(Claim)

Take now $(\underline{U}, \underline{V}) := (0, V)$, it can easily be checked that this function is the generalised sub-solution promised by lemma 35 whom the proof is therefore achieved. $\square$

Proof (Theorem 34) $\boxed{\textbf{Too fast viewer}}$
Assume $c > C_*$ and consider as defined in section II.6 the exponential super-solution of problem (II.2)

$$\begin{pmatrix} \overline{u}(t,x) \\ \overline{v}(t,x,y) \end{pmatrix} \overset{\text{def}}{=} \begin{pmatrix} e^{\alpha(x+C_*t)} \\ \gamma e^{\alpha(x+C_*t)-\beta y} \end{pmatrix}.$$

By an horizontal symmetry in space, the function

$$\begin{pmatrix} \widetilde{u}(t,x) \\ \widetilde{v}(t,x,y) \end{pmatrix} := \begin{pmatrix} e^{\alpha(-x+C_*t)} \\ \gamma e^{\alpha(-x+C_*t)-\beta y} \end{pmatrix}$$

is also a super-solution. Then, up to take for some positive t_0

$$\left(\overline{u}(t_0+t,x), \overline{v}(t_0+t,x,y)\right) \text{ instead of } \left(\overline{u}(t,x), \overline{v}(t,x,y)\right)$$

and

$$\left(\widetilde{u}(t_0+t,x), \widetilde{v}(t_0+t,x,y)\right) \text{ instead of } \left(\widetilde{u}(t,x), \widetilde{v}(t,x,y)\right),$$

it is not restrictive to suppose that $(\overline{u}, \overline{v})$ and $(\widetilde{u}, \widetilde{v})$ are over the initial datum (u_0, v_0) because the latter is bounded and compactly supported. Whence, thanks to the comparison principle, we get

$$(u,v) \leq (\overline{u}, \overline{v}) \qquad \text{and} \qquad (u,v) \leq (\widetilde{u}, \widetilde{v})$$

everywhere and at any time. We have

$$(0,0) \overset{\text{CP}}{\leq} \sup_{|x|\geq ct} \left(u(t,x), v(t,x,y)\right) \leq \underbrace{\sup_{x\geq ct}\left(u(t,x), v(t,x,y)\right)}_{(\bigstar)} + \underbrace{\sup_{x\leq -ct}\left(u(t,x), v(t,x,y)\right)}_{(\clubsuit)}.$$

Let's prove that both ($\bigstar$) and ($\clubsuit$) become $(0,0)$ as t tends to infinity.

$$\begin{aligned}(\bigstar) &= \sup_{x\geq ct}\left(u\left(t,x\right),v\left(t,x,y\right)\right)\\ &\overset{\text{CP}}{\leq} \sup_{x\geq ct}\left(\widetilde{u}\left(t,x\right),\widetilde{v}\left(t,x,y\right)\right)\\ &= \sup_{x\geq ct}\left(e^{\alpha(-x+C_*t)},\gamma e^{\alpha(-x+C_*t)-\beta y}\right)\\ &= \left(e^{\alpha(-ct+C_*t)},\gamma e^{\alpha(-ct+C_*t)-\beta y}\right)\\ &= \left(e^{\alpha(C_*-c)t},\gamma e^{\alpha(C_*-c)t-\beta y}\right)\end{aligned}$$

then, because by assumption $C_*-c<0$,

$$(\bigstar)\overset{t\to\infty}{\longrightarrow}(0,0).$$

Proving that ($\clubsuit$) $\overset{t\to\infty}{\longrightarrow}$ $(0,0)$ follows the same mood:

$$\begin{aligned}(\clubsuit) &= \sup_{x\leq -ct}\left(u\left(t,x\right),v\left(t,x,y\right)\right)\\ &\overset{\text{CP}}{\leq} \sup_{x\leq -ct}\left(\overline{u}\left(t,x\right),\overline{v}\left(t,x,y\right)\right)\\ &= \sup_{x\leq -ct}\left(e^{\alpha(x+C_*t)},\gamma e^{\alpha(x+C_*t)-\beta y}\right)\\ &= \left(e^{\alpha(-ct+C_*t)},\gamma e^{\alpha(-ct+C_*t)-\beta y}\right)\\ &= \left(e^{\alpha(C_*-c)t},\gamma e^{\alpha(C_*-c)t-\beta y}\right)\\ &\overset{t\to\infty}{\longrightarrow}(0,0).\end{aligned}$$

Therefore, one finally gets the wanted result: $\lim_{t\to\infty}\left(\sup_{|x|\geq ct}\left(u\left(t,x\right),v\left(t,x,y\right)\right)\right)=(0,0)$. □

We prove now the second part of theorem 34 which is the less simple of the two.

Proof (Theorem 34) **Too slow viewer**

Assume $c\in(0;C_*)=(0;C_{\text{KPP}})$ and consider the stationary generalised sub-solution $(\underline{u},\underline{v})$ of problem (II.16) provided by lemma 35 (note we take c close enough to C_* if $D>2d$ to ensure the existence of such sub-solution).

Claim. There exists γ_0 such that for all $\gamma\in(0;\gamma_0]$ the couple $(\gamma\underline{u},\gamma\underline{v})$ is a sub-solution of the following non-linear problem in the moving framework

$$\begin{cases}\partial_t u - D\partial_{xx}u + c\partial_x u = v\left(t,x,0\right)-\mu u & (t,x,0)\in(0;\infty)\times\mathcal{R}\\ \partial_t v - d\Delta v + c\partial_x v = f\left(v\right) & (t,x,y)\in(0;\infty)\times\mathcal{F}\\ -d\partial_y v\left(t,x,0\right)=\mu u\left(t,x\right)-v\left(t,x,0\right) & (t,x,0)\in(0;\infty)\times\mathcal{R}.\end{cases}\qquad\textbf{(II.20)}$$

Let's prove that claim. Showing that $(\gamma\underline{u}, \gamma\underline{v})$ is a sub-solution for first and third line of problem (II.20) does not pose any problem if one reminds that $(\underline{u}, \underline{v})$ is a sub-solution of (II.16); indeed, we have well

$$\begin{aligned}\partial_t(\gamma\underline{u}) - D\partial_{xx}(\gamma\underline{u}) + &c\partial_x(\gamma\underline{u}) - \gamma\underline{v}(t,x,0) + \mu\gamma\underline{u} \\ &= \gamma\left(\partial_t\underline{u} - D\partial_{xx}\underline{u} + c\partial_x\underline{u} - \underline{v}(t,x,0) + \mu\underline{u}\right) \\ &\leq 0,\end{aligned}$$

and

$$\begin{aligned}-d\partial_y(\gamma\underline{v})(t,x,0) - &\mu\gamma\underline{u}(t,x) + \gamma\underline{v}(t,x,0) \\ &= \gamma\left(-d\partial_y\underline{v}(t,x,0) - \mu\underline{u}(t,x) + \underline{v}(t,x,0)\right) \\ &\leq 0.\end{aligned}$$

It thus remains to show that $(\gamma\underline{u}, \gamma\underline{v})$ is a sub-solution for second line of problem (II.20). One has

$$\begin{aligned}\partial_t(\gamma\underline{v}) - d\Delta(\gamma\underline{v}) + c\partial_x(\gamma\underline{v}) &= \gamma\left(\partial_t\underline{v} - d\Delta\underline{v} + c\partial_x\underline{v}\right) \\ &\leq \gamma\left(f'(0) - \delta\right)\underline{v} \\ &= \left(f'(0) - \delta\right)(\gamma\underline{v}). \qquad \textbf{(II.21)}\end{aligned}$$

Then by using Taylor-Young's formula, one gets

$$f(\gamma\underline{v}) = (\gamma\underline{v})f'(0) - \psi(\gamma\underline{v}), \qquad \textbf{(II.22)}$$

where

- $\psi(\gamma\underline{v}) = o\left((\gamma\underline{v})^2\right)$ (Taylor-Young),
- $\psi(\gamma\underline{v}) \geq 0$ (KPP-hypothesis).

Whence, because $\underline{v}$ is bounded, for all $\delta > 0$ and γ small enough, $(\gamma\underline{v})$ is also small enough to get (by comparing quadratic *versus* linear decadence)

$$0 \leq \psi(\gamma\underline{v}) \leq \delta(\gamma\underline{v}).$$

Therefore,

$$-\psi(\gamma\underline{v}) \geq -\delta(\gamma\underline{v})$$

and so, using (II.22),

$$f(\gamma\underline{v}) \geq \left(f'(0) - \delta\right)(\gamma v).$$

Taking up inequality (II.21), one achieves

$$\partial_t(\gamma\underline{v}) - d\Delta(\gamma\underline{v}) + c\partial_x(\gamma\underline{v}) \leq f(\gamma\underline{v}). \qquad \textbf{(II.23)}$$

Hence it has been shown that $(\gamma\underline{u}, \gamma\underline{v})$ is a sub-solution of whole problem (II.20). One draws the attention of the reader on the fact that inequality (II.23) is actually strict in every point where $\underline{v}$ is non-zero and so, as such points exist, $(\gamma\underline{u}, \gamma\underline{v})$ cannot be a solution of (II.20). $\square$(Claim)

Let thus $\gamma \in (0; \gamma_0]$, we denote by (u_γ, v_γ) the solution of the non-linear problem (II.20) starting from the initial datum $(\gamma\underline{u}, \gamma\underline{v})$. Because, thanks to the claim, $(u_\gamma, v_\gamma)_{|t=0} = (\gamma\underline{u}, \gamma\underline{v})$ is a sub-solution of the problem whom it is solution, by some similar arguments as those used to show corollary 19, the function (u_γ, v_γ) is increasing with respect to the time when space is fixed. Moreover, because $(\gamma\underline{u}, \gamma\underline{v})$ is not a solution of (II.20), $(\gamma\underline{u}, \gamma\underline{v})$ is strictly bellow $(u_\gamma(t, \bullet), v_\gamma(t, \bullet))$ for all positive time t. Indeed:

- $(\gamma\underline{u}, \gamma\underline{v})$ is a sub-solution,
- (u_γ, v_γ) is a solution and then also a super-solution,
- $(\gamma\underline{u}, \gamma\underline{v}) \leq (u_\gamma(0, \bullet), v_\gamma(0, \bullet))$,

whence by applying the generalised comparison principle,

- either exists $T > 0$ such that $(\gamma\underline{u}, \gamma\underline{v}) = (u_\gamma(t, \bullet), v_\gamma(t, \bullet))$ in $[0; T]$,
- or $(\gamma\underline{u}, \gamma\underline{v}) < (u_\gamma(t, \bullet), v_\gamma(t, \bullet))$ for all positive time t.

First situation have to be discarded otherwise $(\gamma\underline{u}, \gamma\underline{v})$ would be a solution of (II.20) and one thereby gets the wanted conclusion.

The increase of bounded functions $(u_\gamma(\bullet, x), v_\gamma(\bullet, x, y))$ ($(x, y) \in \mathcal{F}$) allows us to say that (u_γ, v_γ) converges pointwise in space toward a limit function (U_γ, V_γ) as $t \to \infty$. Furthermore, by parabolic estimates, that convergence happens (up to a sub-sequence) locally uniformly in space. Now, as (u_γ, v_γ) is strictly above its initial datum for all positive time, we have the strict order relation $(U_\gamma, V_\gamma) > (\gamma\underline{u}, \gamma\underline{v})$ and one thus may find some $k > 0$ such that for all $h \in (-k; k)$,

$$\left(\gamma\underline{u}^{(h)}, \gamma\underline{v}^{(h)}\right) < (U_\gamma, V_\gamma),$$

where $\left(\gamma\underline{u}^{(h)}, \gamma\underline{v}^{(h)}\right)$ denotes $(\gamma\underline{u}, \gamma\underline{v})$ translated of h along the x-direction. Call $\left(u_\gamma^{(h)}, v_\gamma^{(h)}\right)$ the solution of (II.20) starting from the initial datum $\left(\gamma\underline{u}^{(h)}, \gamma\underline{v}^{(h)}\right)$, the last inequality spreads then by comparison for all positive time, that is

$$\left(u_\gamma^{(h)}, v_\gamma^{(h)}\right) < (U_\gamma, V_\gamma).$$

Take then the limit as $t \to \infty$, one obtains

$$\left(U_\gamma^{(h)}, V_\gamma^{(h)}\right) \leq (U_\gamma, V_\gamma).$$

So (U_γ, V_γ) is above its translated along the x-direction and therefore that couple has to be independent of the x variable; that is

- $U_\gamma \equiv$ constant, and

- $V_\gamma = V_\gamma(y)$.

Because of the local uniform in space convergence of (u_γ, v_γ) toward (U_γ, V_γ), the latter satisfies the stationary version of problem (II.20) which gives, combined with the independence to x, the following ODE Cauchy problem:

$$\begin{cases} V_\gamma(0) = \mu U_\gamma \\ V_\gamma'' = -f(V_\gamma)/d \quad \text{in } \mathbb{R}_+^* \\ V_\gamma'(0) = 0. \end{cases} \qquad \textbf{(II.24)}$$

By posing $X := \begin{pmatrix} V_\gamma \\ V_\gamma' \end{pmatrix}$, one may rewrite that problem by the following way:

$$\begin{cases} X' = F(X) \quad \text{in } \mathbb{R}_+^* \\ X(0) = \begin{pmatrix} \mu U_\gamma \\ 0 \end{pmatrix}. \end{cases} \qquad \textbf{(II.25)}$$

where $U_\gamma \in (0; 1/\mu]$ plays as a parameter and the function F is smooth and defined by

$$F(X_1, X_2) := (X_2, -f(X_1)/d).$$

Existence and uniqueness of a solution for (II.25) is then guaranteed thanks to the Cauchy-Lipschitz theorem, and one easily can see that F owns two equilibrium points which are

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

One distinguish then two cases depending on the value of the parameter U_γ:

- if $U_\gamma \equiv 1/\mu$, then $X := \begin{pmatrix} V_\gamma \\ V_\gamma' \end{pmatrix}$ starts from the equilibrium point $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so in that case $V_\gamma \equiv 1$,
- if $U_\gamma \equiv \ell \in (0; 1/\mu)$, we have

$$V_\gamma''(0) = -\frac{f(V_\gamma(0))}{d} = -\frac{f(\ell)}{d} < 0,$$

then for small $y > 0$, $V_\gamma'(y) < 0$ and $V_\gamma(y) \in (0; 1)$ and so $V_\gamma''(y)$ remains negative. Whence V_γ is concave in y which implies that V_γ is bellow its tangent at point y and whom the slope is negative. Thereby V_γ has to hit 0 for some positive y_0 which is exclude because of the order relation $(0,0) \leq (\gamma \underline{u}, \gamma \underline{v}) < (U_\gamma, V_\gamma)$. This case has therefore to be discarded.

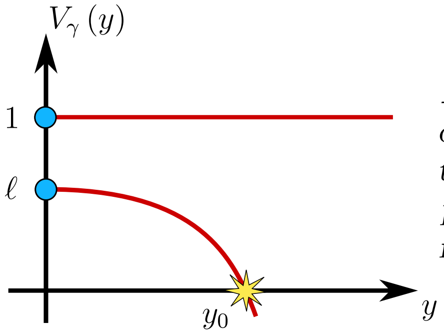


Figure F33 – *Illustration of the two possible cases depending on the value of U_γ. One sees here that the properties "$U_\gamma \equiv \ell \in (0; 1/\mu)$" and "$V_\gamma$ remains positive" are not compatible; therefore U_γ has to be identically equal to $1/\mu$ and then $V_\gamma \equiv 1$.*

Hence the conclusion we obtain *via* this ODE reasoning is that $(U_\gamma, V_\gamma) \equiv (1/\mu, 1)$.

To achieve the proof, we are now going to compare the solution (u, v) with the functions

$$\begin{pmatrix} u_\gamma\,(t, x+ct) \\ v_\gamma\,(t, x+ct, y) \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} u_\gamma\,(t, -x+ct) \\ v_\gamma\,(t, -x+ct, y) \end{pmatrix}. \tag{\textbf{II.26}}$$

Note that the initial Field-Road problem (II.2) and the problem in the moving framework (II.20) are actually equivalent in the following way:

$$\begin{gathered} \left[\left(u\,(\bullet,\bullet)\,, v\,(\bullet,\bullet,\bullet)\right) \text{ is solution of (II.2)}\right] \\ \Updownarrow \\ \left[\left(u\,(\bullet,\bullet-ct)\,, v\,(\bullet,\bullet-ct,\bullet)\right) \text{ is solution of (II.20)}\right]. \end{gathered}$$

Therefore, both functions given in (II.26) are some solutions of (II.2).

Up to let a bit of time elapse, it is not restrictive to suppose that $(u, v)_{|t=0}$ is positive and then, up to choosing some smaller $\gamma \in (0;\gamma_0]$, one may get $(\gamma\underline{u}, \gamma\underline{v}) < (u, v)_{|t=0}$. Take now $T > 2$ and $0 \leq \xi \leq c\,(T-2)$. One sets then $\tau \in [1; T/2]$ such that $\xi = c\,(T-2\tau)$. By comparison, we have, for all $t > 0$ and $(x, y) \in \mathcal{F}$,

$$\left(u\,(t,x)\,, v\,(t,x,y)\right) \geq \left(u_\gamma\,(t, x+ct)\,, v_\gamma\,(t, x+ct, y)\right).$$

Then in particular, at time $t = \tau$,

$$\left(u\,(\tau,x)\,, v\,(\tau,x,y)\right) \geq \left(u_\gamma\,(\tau, x+c\tau)\,, v_\gamma\,(\tau, x+c\tau, y)\right)$$

and reminding that (u_γ, v_γ) is increasing with respect to the time,

$$\left(u\,(\tau,x)\,, v\,(\tau,x,y)\right) \geq \left(u_\gamma\,(1, x+c\tau)\,, v_\gamma\,(1, x+c\tau, y)\right). \tag{\textbf{II.27}}$$

One takes now $\eta \in (0;\gamma_0]$ small enough as well (independently of T) in such a way that

$$\left(\eta\underline{u}\,(-x)\,, \eta\underline{v}\,(-x,y)\right) \leq \left(u_\gamma\,(1,x)\,, v_\gamma\,(1,x,y)\right). \tag{\textbf{II.28}}$$

By gathering (II.27) and (II.28) there comes

$$\left(u\,(\tau,x)\,, v\,(\tau,x,y)\right) \geq \left(\eta\underline{u}\,(-x-c\tau)\,, \eta\underline{v}\,(-x-c\tau,y)\right).$$

Applying the comparison principle to that inequality, one obtains

$$\left(u\,(t+\tau,x)\,, v\,(t+\tau,x,y)\right) \geq \left(u_\eta\,(t, -x-c\tau+ct)\,, v_\eta\,(t, -x-c\tau+ct, y)\right).$$

Take $t-\tau$ instead of t,

$$\left(u\,(t,x)\,, v\,(t,x,y)\right) \geq \left(u_\eta\,(t-\tau, -x+c\,(t-2\tau))\,, v_\eta\,(t-\tau, -x+c\,(t-2\tau)\,, y)\right)$$

and assess that at time $t = \tau$,

$$\left(u\,(\tau,x)\,, v\,(\tau,x,y)\right) \geq \left(u_\eta\,(0, -x-c\tau)\,, v_\eta\,(0, -x-c\tau, y)\right).$$

One applies once again the comparison principle on the latter equality:

$$(u(t+\tau,x), v(t+\tau,x,y)) \geq (u_\eta(t,-x+c(t-\tau)), v_\eta(t,-x+c(t-\tau),y)).$$

And take that at time $t = T - \tau$:

$$(u(T,x), v(T,x,y)) \geq (u_\eta(T-\tau,-x+c(T-2\tau)), v_\eta(T-\tau,-x+c(T-2\tau),y)),$$

that is

$$(u(T,x), v(T,x,y)) \geq (u_\eta(T-\tau,-x+\xi), v_\eta(T-\tau,-x+\xi,y)).$$

If one assess that inequality at $x = \xi$, we get

$$(u(T,\xi), v(T,\xi,y)) \geq (u_\eta(T-\tau,0), v_\eta(T-\tau,0,y)).$$

Next, using the time-increase of (u_η, v_η),

$$\begin{aligned}(u(T,\xi), v(T,\xi,y)) &\geq \left(u_\eta\left(\frac{T}{2},0\right), v_\eta\left(\frac{T}{2},0,y\right)\right) \\ &\xrightarrow{t\to\infty} (U_\gamma, V_\gamma) \\ &\equiv \left(\frac{1}{\mu},1\right).\end{aligned}$$

Thereby, one has prove here that

$$\lim_{T\to\infty}\left(\inf_{0\leq\xi\leq c(T-2)}(u(T,\xi), v(T,\xi,y))\right) \geq \left(\frac{1}{\mu},1\right).$$

To expand symmetrically the domain of the infimum in the latter limit, repeat the previous operations by replacing ξ with $-\xi$ and taking τ such that $\xi = -c(T-2\tau)$. By doing this one gets

$$\lim_{T\to\infty}\left(\inf_{|\xi|\leq c(T-2)}(u(T,\xi), v(T,\xi,y))\right) \geq \left(\frac{1}{\mu},1\right).$$

If this inequality would be strict, we would then have in the centred unit ball:

$$\lim_{T\to\infty}\left(\inf_{|\xi|\leq 1}(u(T,\xi), v(T,\xi,y))\right) \geq \lim_{T\to\infty}\left(\inf_{|\xi|\leq c(T-2)}(u(T,\xi), v(T,\xi,y))\right) > \left(\frac{1}{\mu},1\right)$$

which is impossible because (u,v) tends toward $(1/\mu, 1)$ locally uniformly in space. Hence the inequality was actually an equality:

$$\lim_{T\to\infty}\left(\inf_{|\xi|\leq c(T-2)}(u(T,\xi), v(T,\xi,y))\right) = \left(\frac{1}{\mu},1\right),$$

that's the result we were aiming for. □

II.8 C_* behaviour as D tends to $+\infty$

One wonders here how becomes the evolution of $C_* = C_*(\mu, d, D)$ when the Road diffusion D becomes large. For this section, it is assumed that the reader is familiar with the figures and concepts discussed in section II.6 because it is the starting point for the asymptotic reasoning that we are about to do. We announce now the main result we aim to prove:

Theorem 36 (Berestycki *et al.*) (Spreading speed for large Road diffusion)
Let d and μ be fixed positive and D evolving in $(0; \infty)$, one considers the asymptotic spreading speed $C_* = C_*(\mu, d, D)$ given in the previous section ([a]). The following assertion holds then true:

$$\lim_{D\to\infty} C_*/\sqrt{D} \text{ exists and is a real positive number.}$$

Otherwise said,

$$\exists \lambda > 0 \text{ such that } C_* \overset{D\to\infty}{\sim} \lambda\sqrt{D}.$$

[a] Namely, see theorem 34 page 65.

Proving theorem 36 shall require that lemma:

Lemma 37
We are provided with that asymptotic control on C_*^2/D:

$$\sqrt{4\mu^2 + (f'(0))^2} - 2\mu \leq \liminf_{D\to\infty} \frac{C_*^2}{D} \leq \limsup_{D\to\infty} \frac{C_*^2}{D} \leq f'(0). \qquad \textbf{(II.29)}$$

Proof (Lemma 37)
Let $D > 2d$, we are so in the first case of figure (*F31*) page 62:

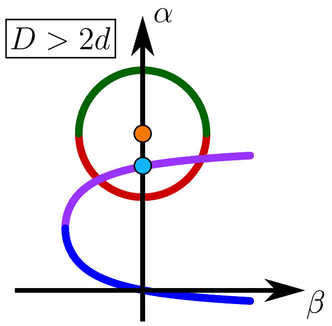


Figure F34 – *First case of figure (F31).*

When $c = C_*$, we have sawn that $\Gamma^-_{c,d}$ and $\Gamma^+_{c,D}$ are tangent and one notes (β_*, α_*) the intersection point between theses curves. We recall that $\Gamma^-_{c,d}$ and $\Gamma^+_{c,D}$ are respectively the graph of the functions

$$\alpha_d^-(c,\beta) := \frac{c - \sqrt{c^2 - C_{\mathrm{KPP}}^2 - 4d^2\beta^2}}{2d}$$

and

$$\alpha_D^+(c,\beta) := \frac{1}{2D}\left(c + \sqrt{c^2 + \frac{4\mu d D\beta}{1+d\beta}}\right).$$

We thus may have $\alpha_d^-(C_*,\beta_*) = \alpha_D^+(C_*,\beta_*)$ and $\partial_\beta \alpha_d^-(C_*,\beta_*) = \partial_\beta \alpha_D^+(C_*,\beta_*)$; and because

- $\partial_\beta \alpha_d^-(C_*,\beta)$ is positive for all β and
- $\partial_\beta \alpha_D^+(C_*,\beta)$ is non-positive for all non-positive β,

β_* cannot be non-positive and we necessarily get $\beta_* > 0$. Furthermore, thanks to the convexity of $\alpha_d^-(C_*,\bullet) - \alpha_D^+(C_*,\bullet)$, $\Gamma^+_{C_*,D}$ is strictly bellow $\Gamma^-_{C_*,D}$ except at the point (β_*,α_*). Translating that geometric affirmation at the point $\beta = 0\,(\neq \beta_*)$, we obtain

$$\begin{aligned}\alpha_D^+(C_*,0) &= \frac{C_*}{D}\\ &< \frac{C_*}{2d} - \beta_{\mathrm{KPP}}(C_*)\\ &= \alpha_d^-(C_*,0). \qquad \textbf{(II.30)}\end{aligned}$$

Finally, because $\alpha_d^-(C_*,\bullet)$ and $\alpha_D^+(C_*,\bullet)$ are increasing as $\bullet$ is positive, we get

$$\begin{aligned}\alpha_d^-(C_*,0) &< \alpha_d^-(C_*,\beta_*)\\ &= \alpha_D^+(C_*,\beta_*)\\ &< \lim_{\beta\to\infty} \alpha_D^+(C_*,\beta)\\ &= \frac{1}{2D}\left(C_* + \sqrt{C_*^2 + 4\mu D}\right). \qquad \textbf{(II.31)}\end{aligned}$$

By bringing (II.30) and (II.31) together we obtain so

$$\frac{C_*}{D} < \frac{C_*}{2d} - \beta_{\mathrm{KPP}}(C_*) < \frac{1}{2D}\left(C_* + \sqrt{C_*^2 + 4\mu D}\right) \qquad \textbf{(II.32)}$$

as you may see on figure (*F35*).

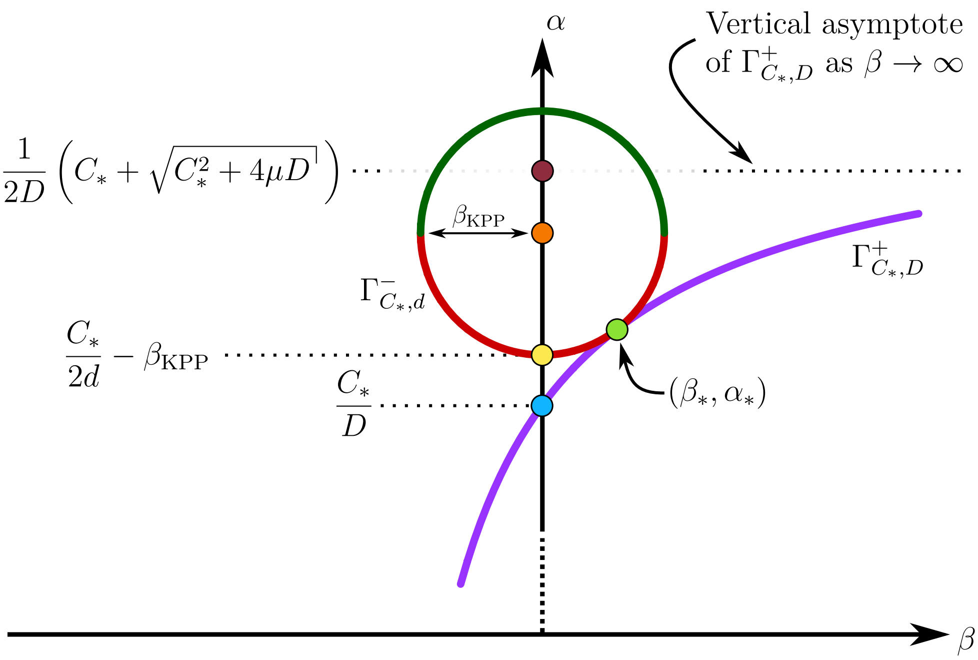


Figure F35 – *Illustration of the inequality (II.32)* which geometrically interprets as "blue is bellow yellow which is bellow dark red".

Recalling that $\beta_{\mathrm{KPP}}(c) := \frac{\sqrt{c^2 - C_{\mathrm{KPP}}^2}}{2d}$, we get

$$\frac{1}{D} < \frac{1}{2d}\left(1 - \sqrt{1 - \frac{C_{\mathrm{KPP}}^2}{C_*^2}}\right) < \frac{1}{2D}\left(1 + \sqrt{1 + \frac{4\mu D}{C_*^2}}\right) \qquad (\textbf{II.33}).$$

Suppose that $[D \to \infty] \not\Rightarrow [C_* \to \infty]$. Then by taking the limit in the second inequality of (II.33), we would have

$$0 < \frac{1}{2d}\left(1 - \sqrt{1 - \frac{C_{\mathrm{KPP}^2}}{\inf\limits_{D\to\infty} C_*^2}}\right) < \frac{1}{2D}\left(1 + \sqrt{1 + \frac{4\mu D}{C_*^2}}\right) \xrightarrow{D\to\infty} 0,$$

that is absurd so $D \to \infty$ implies $C_* \to \infty$. This allows us to use Taylor's series for $\sqrt{1 - \bullet}$ in a vicinity from 0:

$$\sqrt{1 - \frac{C_{\mathrm{KPP}}}{C_*^2}} = 1 - \frac{C_{\mathrm{KPP}}}{2C_*^2} + o\left(\frac{1}{C_*^2}\right).$$

Whence, taking up (II.33),

$$\frac{1}{D} < \frac{1}{2d}\left(\frac{C_{\mathrm{KPP}}^2}{2C_*^2} + o\left(\frac{1}{C_*^2}\right)\right) < \frac{1}{2D}\left(1 + \sqrt{1 + \frac{4\mu D}{C_*^2}}\right)$$

$$\frac{1}{D} < \frac{C_*^{-2}}{2d}\left(\frac{C_{\mathrm{KPP}}^2}{2} + o(1)\right) < \frac{1}{2D}\left(1 + \sqrt{1 + \frac{4\mu D}{C_*^2}}\right) \qquad (\textbf{II.34})$$

Let us take $\tau := C_*/\sqrt{D}$. Then first inequation of (II.34) becomes

$$\tau^2 < \frac{C_{\mathrm{KPP}}^2}{4d} + o(1) = f'(0) + o(1). \qquad \textbf{(II.35)}$$

Take the limit as $D \to \infty$ to get a first piece of the result:

$$\limsup_{D\to\infty} \frac{C_*}{D} \leq f'(0).$$

We work now on second inequality of (II.34),

$$\frac{1}{d}\left(\frac{C_{\mathrm{KPP}}^2}{2} + o(1)\right) < \frac{C_*^2}{D}\left(1 + \sqrt{1 + \frac{4\mu D}{C_*^2}}\right)$$

$$2\frac{C_{\mathrm{KPP}}^2}{4d} + o(1) < \tau^2\left(1 + \sqrt{\frac{1}{\tau^2}(\tau + 4\mu)}\right)$$

$$2f'(0) + o(1) < \tau^2 + \tau\sqrt{(\tau + 4\mu)},$$

and then with (II.35),

$$2f'(0) + o(1) < \tau^2 + \tau\sqrt{(\tau + 4\mu)} < f'(0) + \tau\sqrt{(\tau + 4\mu)} + o(1). \qquad \textbf{(II.36)}$$

Consider the both extremities hands of (II.36), it implies successively

$$f'(0) + o(1) < \tau\sqrt{\tau^2 + 4\mu}$$

$$(f'(0))^2 + o(1) < \tau^2\left(\tau^2 + 4\mu\right)$$

$$(f'(0))^2 + o(1) + 4\mu^2 < \tau^4 + 4\mu\tau^2 + 4\mu^2$$

$$(f'(0))^2 + 4\mu^2 + o(1) < \left(\tau^2 + 2\mu\right)^2$$

$$\sqrt{(f'(0))^2 + 4\mu^2 + o(1)} < \tau^2 + 2\mu$$

$$\sqrt{(f'(0))^2 + 4\mu^2 + o(1)} - 2\mu < \tau^2.$$

Whence, taking the limit as $D \to \infty$, one finally achieves

$$\sqrt{(f'(0))^2 + 4\mu^2 + o(1)} - 2\mu < \liminf_{D\to\infty} \frac{C_*}{D},$$

and thereby the aimed control is shown. □

Proof (II.36) (Spreading speed for large Road diffusion)
We start by reminding the algebraic system (II.15) with which C_* has been obtained:

$$\begin{cases} -D\alpha^2 + c\alpha = \gamma - \mu \\ -d\alpha^2 + c\alpha = f'(0) + d\beta^2 \\ d\beta\gamma = \mu - \gamma. \end{cases} \qquad \text{(II.15)}$$

Lemma 37 gives actually a D-independent asymptotic control on C_*^2/D; this allow us to rescale α and c in (II.15) in the following way: let us take

$$\widetilde{c} := \frac{c}{\sqrt{D}} \qquad \text{and} \qquad \widetilde{\alpha} := \sqrt{D}\,\alpha.$$

We get the rescaled system

$$\begin{cases} -\widetilde{\alpha}^2 + \widetilde{c}\widetilde{\alpha} = \gamma - \mu & (\mathbf{i}) \\ -\frac{d}{D}\widetilde{\alpha}^2 + \widetilde{c}\widetilde{\alpha} = f'(0) + d\beta^2 & (\mathbf{ii}) \\ d\beta\gamma = \mu - \gamma. & (\mathbf{iii}) \end{cases} \qquad (\mathbf{II.37})$$

As we said above, thanks to lemma 37, the amount $\widetilde{C}_* := C_*/\sqrt{D}$ remains bounded independently of D as this one tends to infinity – note however that it is not yet sure that $C_*/\sqrt{D}$ has a limit. More precisely, the lower bound of $\widetilde{C}_*$ stay away from 0 and the upper one is bellow $f'(0)$. Looking at (**i**) in (II.37), one sees that $\widetilde{\alpha}$ has also to be bounded as $D \to \infty$ (otherwise we would get $-\infty = \gamma - \mu$); so by taking the limit of system (II.37), the amount $-\frac{d}{D}\widetilde{\alpha}^2$ vanishes and we obtain: ((**i**) and (**iii**) of (II.37) have been gathered in (**i**) of (II.38)

$$\begin{cases} -\widetilde{\alpha}^2 + \widetilde{c}\widetilde{\alpha} = -\frac{d\beta\mu}{1+d\beta} & (\mathbf{i}) \\ \widetilde{\alpha} = \left(f'(0) + d\beta^2\right)/\widetilde{c} & (\mathbf{ii})\,. \end{cases} \qquad (\mathbf{II.38})$$

System (II.38) is the asymptotic version of the one (II.37). When D shall be large, $\widetilde{C}_*$ will tend to be (provided that exists) the minimal $\widetilde{c}$ for which (II.38) has solution. The end of that proof is thus to show the existence of such a minimal $\widetilde{c}$.

Actually (II.37) and (II.38) have same behaviour, that is there exists a critical speed under which there is no solution, upper which there are two solutions and at which there is a unique solution. To see that, take a look at equation (**ii**) of (II.38), it is the one of a parabola whom the lowest point remains at $(0, f'(0)/\widetilde{c})$. We call that conic $\Gamma^{\infty}_{\widetilde{c},d}$. Consider now (**i**) of (II.38). One easily may see that it is represented in the $(\beta, \widetilde{\alpha})$ plane by the curve $\Gamma^{\infty}_{\widetilde{c},D} := \Gamma_{\widetilde{c},D_{|D=1}}$ defined in section II.6.

$\boxed{\text{If } \widetilde{c} \text{ is close to } 0}$ then

- $\Gamma^{\infty}_{\widetilde{c},d}$ lies above the line $\widetilde{\alpha} = f'(0)/\widetilde{c}$ which is as large as one wants, and
- $\Gamma^{\infty}_{\widetilde{c},D}$ lies bellow the line $\widetilde{\alpha} = \frac{1}{2D}\left(\widetilde{c} + \sqrt{\widetilde{c}^2 + 4\mu D}\right)$ which is as small as one wants,

hence $\Gamma^{\infty}_{\widetilde{c},d}$ cannot cross $\Gamma^{\infty}_{\widetilde{c},D}$ as it is drawn on the left of figure (*F36*).

$\boxed{\text{If } \widetilde{c} \text{ tends to } \infty}$ then

- the lowest point of the parabola $\Gamma^{\infty}_{\widetilde{c},d}$ (which is $(0, f'(0)/\widetilde{c})$) tends to $(0,0)$, and
- the intersection of $\Gamma^{\infty}_{\widetilde{c},D}$ with the vertical axis (which is $(0,\widetilde{c})$) tends to $(0,\infty)$,

whence $\Gamma^{\infty}_{\widetilde{c},d}$ have to cross $\Gamma^{\infty}_{\widetilde{c},D}$ at least once as it is shown on the right of figure (*F36*).

Therefore, thanks to the two cases bellow and by a convexity argument, there is a unique well-chosen $\widetilde{c}$ such that the curves $\Gamma^{\infty}_{\widetilde{c},d}$ and $\Gamma^{\infty}_{\widetilde{c},D}$ are tangent and thus intersects exactly once as it is illustrated in the middle of figure (*F36*).

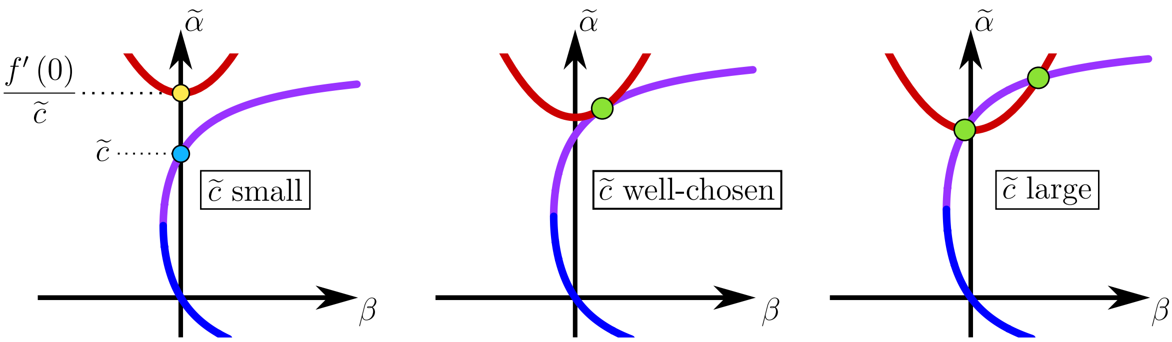


Figure F36 – *Illustration of the three possible cases for system (II.38) which is the D-asymptotic system (II.37).*

According to the notations in the statement of the theorem, call λ that "well-chosen $\widetilde{c}$"; in that way λ is the limit as $D \to \infty$ of $\widetilde{C}_* \overset{\text{def}}{=} C_*/\sqrt{D}$, that completes the proof of the theorem. $\square$

We end that section by numerically checking the result given by theorem (II.36). One actually can assess with a good accuracy the value of $C_* = C_*(\mu, d, D)$ thanks to a dichotomous method for the function $\varphi(c,\beta) := \alpha_d^-(c,\beta) - \alpha_D^+(c,\beta)$.

$\boxed{1}$ Take $C > 0$ large enough to that we are sure that $\Gamma_{c,d}$ and $\Gamma_{c,D}$ intersect twice, and $c = C_{\text{KPP}}$.

$\boxed{2}$ Then the dichotomous loop is the following ($n \in \mathbb{N}^*$ denote the loop number):

- take $\overline{c}_n$ the average of c and C;
- if $\varphi(\overline{c}_n, \bullet)$ remains positive for $\bullet \in \left[0; \frac{1}{2d}\sqrt{\overline{c}_n^2 - C_{\text{KPP}}^2}\right]$, then replace c by $\overline{c}_n$;
- else replace C by $\overline{c}_n$.

$\boxed{3}$ Repeat step $\boxed{2}$ until getting the wanted accuracy. Note the sequence $(c_n)_{n\in\mathbb{N}^*}$ converges exponentially up to C_* thanks to the following equality

$$|\overline{c}_{n+1} - C_*| \leq \frac{C}{2^n}.$$

Finally, in order to numerically check theorem 36, we just have to plot $(D, C_*(\mu, d, D))$ in a "log/log" scale. Because, as asserts the theorem, $C_* \overset{D\to\infty}{\sim} \lambda\sqrt{D}$, we should observe that the points are aligned along a line parallel to $y = x/2$.

For $\mu = d = 1$ fixed, the method given above provides some conclusive result:

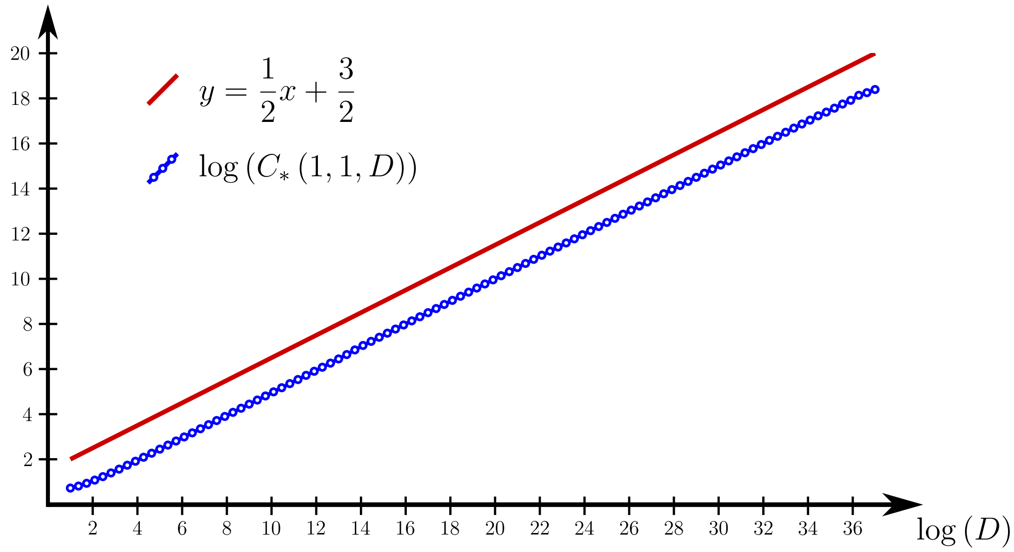


Figure F37 – *Numerical testing of theorem 36 as $\mu = d = 1$. Each ∘ represents an assessment of C_*.*

PART III

Research on the Field-Road space $\mathbb{R}^N_+$

We are willing in this part to induce a weak Allee effect on the reaction f of the Field-Road model proposed by Berestycki *et al.* in [4]:

$$\begin{cases} \partial_t u - D\partial_{xx} u = v\left(t,x,0\right) - \mu u & (t,x,0) \in (0;\infty) \times \mathcal{R} \\ \partial_t v - d\Delta v = f\left(v\right) & (t,x,y) \in (0;\infty) \times \mathcal{F} \\ -d\partial_y v\left(t,x,0\right) = \mu u\left(t,x\right) - v\left(t,x,0\right) & (t,x,0) \in (0;\infty) \times \mathcal{R}. \end{cases} \tag{\textbf{III.1}}$$

More precisely, one would like to take, instead of a logistic reaction as done in [4], a monostable degenerate one, that is

$$f\left(v\right) = v^{1+p}\left(1-v\right)$$

where p denotes a positive constant. Note $p = 0$ returns a logistic reaction.

We have mentioned in section I.5 a few words about monostable degenerate Reaction-Diffusion equations in the whole space $\mathbb{R}^N$ with the Aronson-Weinberger's theorem 23 ([a]) whom the proof ([b]) is mainly based on the Fujita's observation in [7] which is the following: the solution of the semi-linear R-D Cauchy problem

$$\begin{cases} \partial_t v = d\Delta v + v^{1+p} & (t,X) \in (0;\infty) \times \mathbb{R}^N \\ v\left(0,X\right) = v_0\left(X\right) \geq 0 & X \in \mathbb{R}^N \end{cases}$$

is global and extincts if $p > p_F \overset{\text{def}}{=} 2/N$ and blows-up in finite time if $p \leq p_F$.

That Fujita's exponent p_F derives actually from the L^∞ rate of decrease of the heat kernel K on $\mathbb{R}^N$:

$$\left\|K\left(t,\bullet\right)\right\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{t^{N/2}}.$$

Because the heat kernel does not remain the same when space and boundary conditions are changed, we have to work on that aspect in order to obtain a better understanding of it in the case of the Field-Road space $\mathbb{R}^N_+$. When we shall able to assess the L^∞ decay rate of the Road-Field heat kernel, it is expected that it helps us to determine some Field-Road Fujita's exponent with the ambition to draw up an analogous result as Aronson-Weinberger's theorem 23 in the Field-Road framework.

[a] One recalls that this theorem gives a threshold power $p_F \overset{\text{def}}{=} 2/N$ splitting systematic ($p \leq p_F$) and non-systematic ($p > p_F$) Hair Trigger Effect.

[b] That is not given in this report but may be found in [2].

III.1 Heat kernel in the half-space $\mathbb{R}_+$

We work here in one dimension – a generalisation to the N-dimensional case shall be done in the next section. One considers for bounded and non-negative $u_0 : \mathbb{R}_+^* \to \mathbb{R}_+$ the heat equation on the half space $\mathbb{R}_+^*$ provided with initial datum u_0 and Robin boundary conditions:

$$\begin{cases} \partial_t u = \partial_{xx} u & (t,x) \in (0;\infty) \times \mathbb{R}_+^* \\ \alpha u(t,0) - (1-\alpha)\,\partial_x u(t,0) = 0 & t \in (0;\infty) \\ u(0,x) = u_0(x) & x \in \mathbb{R}_+^* \end{cases} \qquad \textbf{(III.2)}$$

where α is a real number in $[0;1]$ and the diffusion coefficient d has been chosen, thanks to some time rescaling, equal to 1.

- $\boxed{\text{Neumann}}$ $\alpha = 0$ provides some *Neumann boundary conditions*: all individuals hitting the frontier $x = 0$ re-bounce inside the domain.
- $\boxed{\text{Dirichlet}}$ $\alpha = 1$ provides some *Dirichlet boundary conditions*: all individuals hitting the frontier $x = 0$ are instantly dead.
- $\boxed{\text{Robin}}$ $0 < \alpha < 1$ provides a mix between Dirichlet and Neumann boundary conditions which are called *Robin boundary conditions* ([c]): an α-proportion of the individuals hitting the frontier $x = 0$ are killed and the others re-bounce inside the domain.

We are now seeking to find the α-Robin-$\mathbb{R}_+^*$-heat kernel $K_\alpha = K_\alpha(t,x,y)$ which shall provide the solutions of Cauchy problem (III.2) for these three different boundary conditions. Note however that the method we shall use to treat the Robin case does not require $\alpha \neq 0$; but we are willing to do the Neumann and Dirichlet cases independently because it is an easier and formative way to approach the problem.

Each of these three methods are based on the section 3 of the book of Strauss [19] whom reasoning is the following:

$\boxed{1}$ one expands the initial datum u_0 on the whole space $\mathbb{R}$ in a way which is specific to the case we treat,

$\boxed{2}$ one solves the problem in the whole space $\mathbb{R}$ for that new expanded initial datum by convoluing the latter to the $\mathbb{R}$-heat kernel,

$\boxed{3}$ one cuts the obtained solution at $x = 0$ and only take the positive part.

For each case we shall furthermore assess the $L^\infty(\mathbb{R})$ norm of the function $K_\alpha(t, \bullet, y)$ whom we want to get a control of the shape c/t^k. We expect, because the individuals leakage is all the more important as α is close to 1, that $k = k(\alpha)$ is increasing (in the broadest sense) with respect to α.

[c] One may also find "*Fourier boundary conditions*"

III.1.1 Neumann boundary conditions

This case is the simplest of the three: one takes $\widetilde{u_0}$ the even-extension of u_0 , that is

$$\widetilde{u_0}(x) := \begin{cases} u_0(x) & \text{if } x \geq 0, \\ u_0(-x) & \text{otherwise.} \end{cases}$$

Note u_0 may not satisfies the Neumann boundary condition at $x = 0$; in that case $\widetilde{u_0}$ cannot be smoother than $\mathcal{C}^0$ in a vicinity of zero. However,

Proposition 38

Let $\widetilde{u} = \widetilde{u}(t, x)$ be the solution of $\partial_t \widetilde{u} = \Delta \widetilde{u}$ starting from $\widetilde{u_0}$, then for all $t > 0$, $\widetilde{u}$ is derivable at $x = 0$ and $\partial_x \widetilde{u}(t, 0) = 0$.

The smoothness result given by that proposition is not surprising and comes from the regularizing effect of the $\mathbb{R}$-heat kernel we have sawn in section I.2. The main contribution provided is then the horizontal slope of $\widetilde{u}$ for all positive time at $x = 0$. That's convenient because we just have to restrict $\widetilde{u}$ to $\{x > 0\}$ in order to get a solution to problem (III.2).

Proof (Proposition 38)

The aim of that proof is therefore showing that $\partial_x \widetilde{u}(t, 0) = 0$ for all $t > 0$; and because $\widetilde{u}(t, \bullet)$ is smooth, it is sufficient to prove that $\widetilde{u}(t, \bullet)$ remains even like its initial datum.

$$\widetilde{u}(t, -x) = \int_{\mathbb{R}} K(t, -x - y)\, \widetilde{u_0}(y)\, \mathrm{d}y$$

take $z = -y$,

$$\widetilde{u}(t, -x) = \int_{\mathbb{R}} K(t, z - x)\, \widetilde{u_0}(-z)\, \mathrm{d}z$$

remind that $K(t, \bullet)$ and $\widetilde{u_0}$ are even,

$$\begin{aligned} \widetilde{u}(t, -x) &= \int_{\mathbb{R}} K(t, x - z)\, \widetilde{u_0}(z)\, \mathrm{d}z \\ &= \widetilde{u}(t, x). \end{aligned}$$

Hence the proof is achieved. □

Take now $u := \widetilde{u}_{|x>0}$, then u is a solution of (III.2) with $\alpha = 0$.

Remark. Uniqueness may be obtained thanks to the comparison principle.

We try now to find the 0-Robin-$\mathbb{R}_+^*$-heat kernel, *i.e.* a *fundamental solution* for problem (III.2), otherwise said, we aim to reach a function $K_0 = K_0(t, x, y)$ such that one can express u under the integral form

$$u\left(t,x\right)=\int_0^{\infty}K_0\left(t,x,y\right)u_0\left(y\right)\mathrm{d}y.$$

Let's try that:

$$\begin{aligned}u\left(t,x\right)&=\int_{\mathbb{R}}K\left(t,x-y\right)\widetilde{u_0}\left(y\right)\mathrm{d}y\\&=\int_{-\infty}^{0}K\left(t,x-y\right)\widetilde{u_0}\left(y\right)\mathrm{d}y+\int_0^{+\infty}K\left(t,x-y\right)u_0\left(y\right)\mathrm{d}y\end{aligned}$$

pose $z=-y$ in first integral,

$$u\left(t,x\right)=\int_0^{+\infty}K\left(t,x+z\right)\widetilde{u_0}\left(-z\right)\mathrm{d}z+\int_0^{+\infty}K\left(t,x-y\right)u_0\left(y\right)\mathrm{d}y$$

remind $\widetilde{u_0}$ is even (and recall z by y),

$$u\left(t,x\right)=\int_0^{+\infty}K\left(t,x+y\right)u_0\left(y\right)\mathrm{d}y+\int_0^{+\infty}K\left(t,x-y\right)u_0\left(y\right)\mathrm{d}y$$

finally, bringing the both integrals in one single,

$$u\left(t,x\right)=\int_0^{\infty}\underbrace{\left[K\left(t,x-y\right)+K\left(t,x+y\right)\right]}_{\text{let us call that } K_0(t,x,y).}u_0\left(y\right)\mathrm{d}y$$

To conclude that case, we look at the $L^{\infty}\left(\mathbb{R}_+\right)$ norm of the kernel $K_0\left(t,\bullet,y\right)$:

$$\left\|K_0\left(t,\bullet,y\right)\right\|_{L^{\infty}(\mathbb{R}_+)}\leq\frac{c}{t^{1/2}},$$

where c only depends on the diffusion coefficient d.

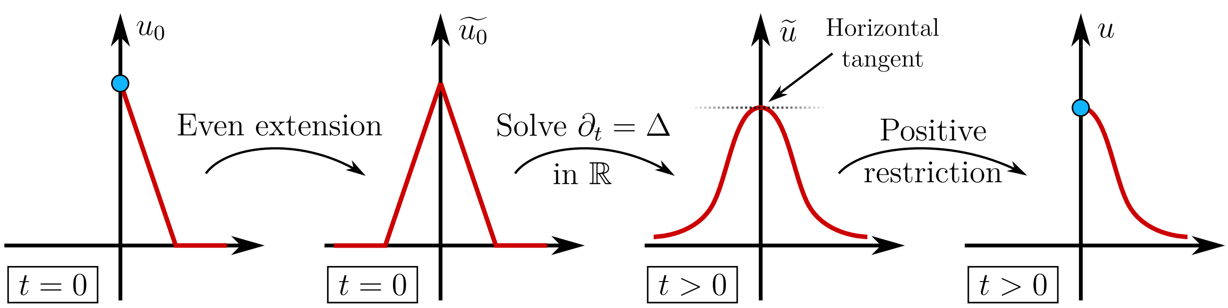


Figure F38 – *Illustration of the steps allowing to find the solution of the heat equation in the half space $\mathbb{R}_+^*$ with Neumann boundary conditions.*

III.1.2 Dirichlet boundary conditions

One takes now $\widetilde{u_0}$ the odd-extension of u_0 , that is

$$\widetilde{u_0}(x) := \begin{cases} u_0(x) & \text{if } x \geq 0, \\ -u_0(-x) & \text{otherwise.} \end{cases}$$

Note u_0 may not be zero at $x = 0$; in that case $\widetilde{u_0}$ cannot be continuous zero. However,

Proposition 39
Let $\tilde{u} = \tilde{u}(t, x)$ be the solution of $\partial_t \tilde{u} = \Delta \tilde{u}$ starting from $\widetilde{u_0}$, then for all $t > 0$, $\tilde{u}(t, 0) = 0$.

The proof of that proposition consists in the same reasoning than the Neumann case:

Proof (Proposition 39)
Because $\tilde{u}(t, \bullet)$ is smooth, in order to show that $\tilde{u}(t, \bullet)$ is zero at $x = 0$, it is sufficient to prove that $\tilde{u}(t, \bullet)$ remains odd like its initial datum.

$$\tilde{u}(t, -x) = \int_{\mathbb{R}} K(t, -x - y)\, \widetilde{u_0}(y)\, \mathrm{d}y$$

take $z = -y$,

$$\tilde{u}(t, -x) = \int_{\mathbb{R}} K(t, z - x)\, \widetilde{u_0}(-z)\, \mathrm{d}z$$

remind that $K(t, \bullet)$ is even and $\widetilde{u_0}$ is odd,

$$\begin{aligned} \tilde{u}(t, -x) &= -\int_{\mathbb{R}} K(t, x - z)\, \widetilde{u_0}(z)\, \mathrm{d}z \\ &= -\tilde{u}(t, x). \end{aligned}$$

Hence the proof is achieved. □

Take now $u := \tilde{u}_{|x>0}$, then u is a solution of (III.2) with $\alpha = 1$.

Remark. Uniqueness may again be obtained thanks to the comparison principle.

We aim now to find the 1-Robin-$\mathbb{R}_+^*$-heat kernel:

$$\begin{aligned} u(t, x) &= \int_{\mathbb{R}} K(t, x - y)\, \widetilde{u_0}(y)\, \mathrm{d}y \\ &= \int_{-\infty}^{0} K(t, x - y)\, \widetilde{u_0}(y)\, \mathrm{d}y + \int_{0}^{+\infty} K(t, x - y)\, u_0(y)\, \mathrm{d}y \end{aligned}$$

pose $z = -y$ in first integral,

$$u(t,x) = \int_0^{+\infty} K(t, x+z)\, \widetilde{u_0}(-z)\, \mathrm{d}z + \int_0^{+\infty} K(t, x-y)\, u_0(y)\, \mathrm{d}y$$

remind $\widetilde{u_0}$ is odd (and recall z by y),

$$u(t,x) = -\int_0^{+\infty} K(t, x+y)\, u_0(y)\, \mathrm{d}y + \int_0^{+\infty} K(t, x-y)\, u_0(y)\, \mathrm{d}y$$

finally, bringing the both integrals in one single,

$$u(t,x) = \int_0^{\infty} \underbrace{\left[K(t, x-y) - K(t, x+y)\right]}_{\text{let us call that } K_1(t,x,y).} u_0(y)\, \mathrm{d}y$$

Finally, looking at the $L^\infty(\mathbb{R}_+)$ norm of the kernel $K_1(t, \bullet, \bullet)$ requires further considerations than for the Neumann case.

Proposition 40

$\boxed{1}$ For all $(t,x,y) \in \mathbb{R}_+^* \times \mathbb{R}_+ \times \mathbb{R}_+$, the amount $K_1(t,x,y)$ remains positive.

$\boxed{2}$ There exists some real positive constant c depending only on d such that, for all $t > 0$,

$$\left\| K_1(t, \bullet, y) \right\|_{L^\infty(\mathbb{R}_+)} \leq \frac{cy}{t}.$$

Proof (Proposition 40) $\boxed{1}$

Let's prove that K_1 is positive: let t be positive and x, y non-negative,

$$\begin{aligned} K_1(t,x,y) &= K(t, x-y) - K(t, x+y) \\ &= \frac{1}{(4\pi t)^{1/2}} \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) \\ &= \underbrace{\frac{e^{-\frac{x^2+y^2}{4t}}}{(4\pi t)^{1/2}}}_{>0} \underbrace{\left(e^{\frac{xy}{2t}} - e^{-\frac{xy}{2t}} \right)}_{>0}. \quad \square \end{aligned}$$

Proof (Proposition 40) $\boxed{2}$

Take $\varphi(y) := e^{-\frac{(x-y)^2}{4t}}$; in that way,

$$K_1(t,x,y) = \frac{1}{(4\pi t)^{1/2}} \left(\varphi(y) - \varphi(-y) \right).$$

We assess $\varphi'(y)$ to get a control on it:

$$\begin{aligned}\varphi'(y) &= \frac{2(x-y)}{4t}e^{-\frac{(x-y)^2}{4t}} \\ &= \frac{1}{\sqrt{t}}\frac{x-y}{2\sqrt{t}}e^{-\left(\frac{x-y}{2\sqrt{t}}\right)^2}\end{aligned}$$

pose $\theta(z) := ze^{-z^2}$,

$$\begin{aligned}\varphi'(y) &= \frac{1}{\sqrt{t}}\cdot\theta\left(\frac{x-y}{2\sqrt{t}}\right) \\ &\leq \frac{1}{\sqrt{t}}\cdot\|\theta\|_{L^\infty} \\ &= \frac{c}{\sqrt{t}}.\end{aligned}$$

By applying now the Mean Value Theorem to function φ, one gets (for some $\xi \in (-y;y)$)

$$\begin{aligned}|\varphi(y)-\varphi(-y)| &= 2y\varphi'(\xi) \\ &\leq \frac{c_1 y}{\sqrt{t}}.\end{aligned}$$

Whence

$$\begin{aligned}K_1(t,x,y) &= \frac{1}{(4\pi t)^{1/2}}\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right) \\ &= \frac{1}{(4\pi t)^{1/2}}\left(\varphi(y)-\varphi(-y)\right) \\ &\leq \frac{c_2 y}{t}.\end{aligned}$$

The desired result follows then from the last equality. $\square$

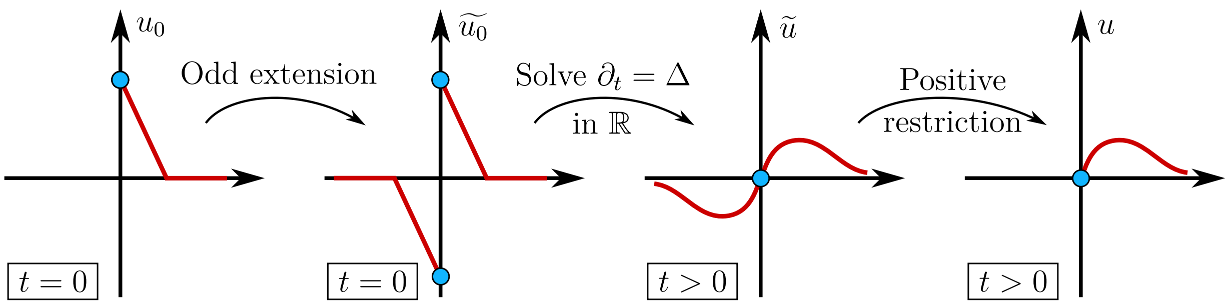


Figure F39 – *Illustration of the steps allowing to find the solution of the heat equation in the half space $\mathbb{R}_+^*$ with Dirichlet boundary conditions.*

III.1.3 Robin boundary conditions

Extend u_0 and express the solution u

Let $\alpha \in [0;1)$, we have now to find the an extension of u_0 for the α-Robin boundary conditions. The common points of the two previous cases was that u_0 was extended such that

- $\widetilde{u_0}'$ was odd in the Neumann case,
- $\widetilde{u_0}$ was odd in the Dirichlet case.

A natural try for the Robin case is therefore choosing $\widetilde{u_0}$ such that

$$\alpha\widetilde{u_0} - (1-\alpha)\,\widetilde{u_0}' \text{ is odd.}$$

Let us state $A := \alpha/\,(1-\alpha)$ so that the latter expanding condition becomes

$$A\widetilde{u_0} - \widetilde{u_0}' \text{ is odd.}$$

That proposition is then equivalent to all the equalities which follow:

$$\begin{aligned}
A\widetilde{u_0}(x) - \widetilde{u_0}'(x) &= -\left(A\widetilde{u_0}(-x) - \widetilde{u_0}'(-x)\right)\\
e^{-Ax}\left(\widetilde{u_0}'(x) - A\widetilde{u_0}(x)\right) &= e^{-Ax}\left(A\widetilde{u_0}(-x) - \widetilde{u_0}'(-x)\right)\\
\left(e^{-Ax}\widetilde{u_0}(x)\right)' &= e^{-Ax}\left(A\widetilde{u_0}(-x) - \widetilde{u_0}'(-x)\right)\\
\widetilde{u_0}(x) &= Ce^{Ax} + e^{Ax}\int_0^x e^{-As}\left(A\widetilde{u_0}(-s) - \widetilde{u_0}'(-s)\right)\mathrm{d}s,
\end{aligned}$$

where C is a constant which should be chosen equal to $u_0(0)$ in order to $\widetilde{u_0}$ keeps its continuity at $x = 0$ [d]. Thereby we pose

$$\widetilde{u_0} := \begin{cases} u_0(x) & \text{if } x \geq 0,\\ u_0(0)\,e^{Ax} + e^{Ax}\int_0^x e^{-As}\left(Au_0(-s) - u_0'(-s)\right)\mathrm{d}s & \text{otherwise}\end{cases}$$

and we reach the solution of the heat equation starting from the initial datum $\widetilde{u_0}$:

$$\widetilde{u}(t,x) := \int_{-\infty}^{+\infty} K(t, x-y)\,\widetilde{u_0}(y)\,\mathrm{d}y.$$

On can by now announce an analogous result as those for the Neumann (proposition 38) and Dirichlet (proposition 39):

Proposition 41
For all $t > 0$, $\widetilde{u}$ is derivable at $x = 0$ and,

$$A\widetilde{u}(t,0) - \partial_x\widetilde{u}(t,0) = 0.$$

[d] Note that the Dirichlet case was the only for which $\widetilde{u_0}$ might not be continuous.

Proof (Proposition 41)
The proof follows the spirit as the Dirichlet's one: we make sure that the oddness of the function $A\widetilde{u_0} - \widetilde{u_0}'$ spreads on $\widetilde{u}$ for all positive time; then because $\widetilde{u}$ and $\partial_x \widetilde{u}$ are smooth, $A\widetilde{u_0} - \widetilde{u_0}'$ has to be zero at $x = 0$, that's the wanted result. Let thus prove that:

$$\begin{aligned} A\widetilde{u}(t,-x) - \partial_x \widetilde{u}(t,-x) &= \int_{\mathbb{R}} K(t,y)\, A\widetilde{u_0}(-x-y)\,\mathrm{d}y - \int_{\mathbb{R}} K(t,y)\,\partial_x\left(\widetilde{u_0}(-x-y)\right)\mathrm{d}y \\ &= \int_{\mathbb{R}} K(t,y)\left[A\widetilde{u_0}(-x-y) - \partial_x\left(\widetilde{u_0}(-x-y)\right)\right]\mathrm{d}y \\ &= \int_{\mathbb{R}} K(t,y)\left[\left(A\widetilde{u_0} - \partial_x\left(\widetilde{u_0}\right)\right)(-x-y)\right]\mathrm{d}y \end{aligned}$$

take $z = -y$,

$$A\widetilde{u}(t,-x) - \partial_x \widetilde{u}(t,-x) = \int_{\mathbb{R}} K(t,-z)\left[\left(A\widetilde{u_0} - \partial_x\left(\widetilde{u_0}\right)\right)(z-x)\right]\mathrm{d}z$$

remind that $A\widetilde{u_0} - \widetilde{u_0}'$ is odd, $K(t,\bullet)$ is even and recall z by y,

$$\begin{aligned} A\widetilde{u}(t,-x) - \partial_x \widetilde{u}(t,-x) &= -\int_{\mathbb{R}} K(t,-y)\left[\left(A\widetilde{u_0} - \partial_x\left(\widetilde{u_0}\right)\right)(x-y)\right]\mathrm{d}y \\ &= -\left(\int_{\mathbb{R}} K(t,y)\, A\widetilde{u_0}(x-y)\,\mathrm{d}y - \int_{\mathbb{R}} K(t,y)\,\partial_x\left(\widetilde{u_0}(x-y)\right)\mathrm{d}y\right) \\ &= -\left(A\widetilde{u}(t,-x) - \partial_x \widetilde{u}(t,-x)\right). \end{aligned}$$

And so the proof is achieved. □

Now we own the solution $\widetilde{u}$ on the whole space $\mathbb{R}$, it just remains to take its restriction to the half positive space to get the solution we want.

Remark. Once again, uniqueness can be obtained thanks to the comparison principle.

Find the heat kernel

We are now seeking to the α-Robin-$\mathbb{R}_+^*$-heat kernel K_α which may be a little bit harder to find than in the Neumann and Dirichlet cases.

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{+\infty} K(t,x-y)\,\widetilde{u_0}(y)\,\mathrm{d}y \\ &= \int_0^{+\infty} K(t,x-y)\,\widetilde{u_0}(y)\,\mathrm{d}y + \int_{-\infty}^{0} K(t,x-y)\,\widetilde{u_0}(y)\,\mathrm{d}y \\ &= \int_0^{+\infty} K(t,x-y)\,u_0(y)\,\mathrm{d}y + \underbrace{\int_0^{+\infty} K(t,x+y)\,\widetilde{u_0}(-y)\,\mathrm{d}y}_{\text{call that integral } I(t,x).} \end{aligned}$$

Let's work on the amount I:

$$
\begin{aligned}
I(t,x) &= \int_0^{+\infty} K(t,x+y)\left(u_0(0)\,e^{-Ay} + e^{-Ay}\int_0^{-y} e^{-As}\left(Au_0(-s) - u_0'(-s)\right)\mathrm{d}s\right)\mathrm{d}y \\
&= \int_0^{+\infty} K(t,x+y)\left(u_0(0)\,e^{-Ay} - e^{-Ay}\int_0^{y} e^{As}\left(Au_0(s) - u_0'(s)\right)\mathrm{d}s\right)\mathrm{d}y \\
&= \int_0^{+\infty} K(t,x+y)\left(u_0(0)\,e^{-Ay} - Ae^{-Ay}\int_0^{y} e^{As}Au_0(s)\,\mathrm{d}s + e^{-Ay}\underbrace{\int_0^{y} e^{As}u_0'(s)\,\mathrm{d}s}_{\text{IBP}}\right)\mathrm{d}y \\
&= \int_0^{+\infty} K(t,x+y)\Bigg(u_0(0)\,e^{-Ay} - Ae^{-Ay}\int_0^{y} e^{As}Au_0(s)\,\mathrm{d}s \\
&\qquad\qquad + u_0(y) - u_0(0)\,e^{-Ay} - Ae^{-Ay}\int_0^{y} e^{As}u_0(s)\,\mathrm{d}s\Bigg)\mathrm{d}y \\
&= \int_0^{+\infty} K(t,x+y)\left(u_0(y) - 2Ae^{-Ay}\int_0^{y} e^{As}u_0(s)\,\mathrm{d}s\right)\mathrm{d}y.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
u(t,x) = \int_0^{+\infty} &\overbrace{\left(K(t,x-y) + K(t,x+y)\right)}^{\text{Recognize here } K_0(t,x,y)} u_0(y)\,\mathrm{d}y \\
&- 2A\int_0^{+\infty}\left(K(t,x+y)\,e^{-Ay}\int_0^{y} e^{As}u_0(s)\,\mathrm{d}s\right)\mathrm{d}y.
\end{aligned}
$$

In the second term of the latter sum, one can check that

$$
K(t,x+y)\,e^{-Ay} = \frac{1}{(4\pi t)^{1/2}}\exp\left(-\frac{\left(y+(2tA+x)\right)^2 - (4tAx + 4d^2t^2A^2)}{4t}\right)
$$

then, by letting $\gamma(z) := e^{-z^2}$, we have

$$
K(t,x+y)\,e^{-Ay} = \frac{e^{Ax+tA^2}}{(4\pi t)^{1/2}}\cdot\gamma\left(\frac{y+(2tA+x)}{2\sqrt{t}}\right).
$$

Therefore,

$$
\begin{aligned}
u(t,x) = \int_0^{+\infty} &K_0(t,x,y)\,u_0(y)\,\mathrm{d}y \\
&- \frac{2Ae^{Ax+tA^2}}{(4\pi t)^{1/2}}\underbrace{\int_0^{+\infty}\left(\gamma\left(\frac{y+(2tA+x)}{2\sqrt{t}}\right)\int_0^{y} e^{As}u_0(s)\,\mathrm{d}s\right)\mathrm{d}y}_{\text{call that } I_1(t,x)\text{, we shall do an IBP on it.}}
\end{aligned}
$$

We pose $\Gamma(z) := -\int_z^{+\infty}\gamma(s)\,\mathrm{d}s$; in that way, Γ is a primitive of γ which is zero when z tends to $+\infty$ and by integrating by part, as specified just above,

$$
I_1(t,x) = -2\sqrt{t}\int_0^{+\infty}\Gamma\left(\frac{y+2tA+x}{2\sqrt{t}}\right)e^{Ay}u_0(y)\,\mathrm{d}y.
$$

So u becomes

$$u\left(t,x\right)=\int_0^{+\infty}K_0\left(t,x,y\right)u_0\left(y\right)\mathrm{d}y+\frac{4A\sqrt{t}}{\left(4\pi t\right)^{1/2}}\underbrace{\int_0^{+\infty}e^{Ax+tA^2+Ay}\cdot\Gamma\left(\frac{y+2tA+x}{2\sqrt{t}}\right)u_0\left(y\right)\mathrm{d}y}_{\text{call that } I_2(t,x)}.$$

Working on I_2:

$$\begin{aligned}I_2\left(t,x\right)=\int_0^{+\infty}&e^{Ax+tA^2+Ay}\cdot\exp\left(-\left(\frac{y+2tA+x}{2\sqrt{t}}\right)^2\right)\\&\cdot\Gamma\left(\frac{y+2tA+x}{2\sqrt{t}}\right)\cdot\exp\left(+\left(\frac{y+2tA+x}{2\sqrt{t}}\right)^2\right)u_0\left(y\right)\mathrm{d}y\\=\int_0^{+\infty}&\exp\left(-\frac{\left(x+y\right)^2}{4t}\right)\cdot\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right)u_0\left(y\right)\mathrm{d}y.\end{aligned}$$

We finally get

$$u\left(t,x\right)=\int_0^{+\infty}\left(K_0\left(t,x,y\right)+4A\sqrt{t}\,K\left(t,x+y\right)\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right)\right)u_0\left(y\right)\mathrm{d}y,$$

consequently we have found the α-Robin-$\mathbb{R}_+^*$-heat kernel:

$$K_\alpha(t,x,y)=K_0\left(t,x,y\right)+4A\sqrt{t}\,K\left(t,x+y\right)\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right).$$

We take the liberty of changing a little bit its form:

$$\begin{aligned}K_\alpha(t,x,y)&=K\left(t,x-y\right)+K\left(t,x+y\right)+4A\sqrt{t}\,K\left(t,x+y\right)\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right)\\&=K\left(t,x-y\right)-K\left(t,x+y\right)+2K\left(t,x+y\right)\\&\qquad+4A\sqrt{t}\,K\left(t,x+y\right)\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right).\end{aligned}$$

Thereby,

$$\boxed{K_\alpha(t,x,y)=K_1\left(t,x,y\right)+2K\left(t,x+y\right)\left(1+2A\sqrt{t}\,\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right)\right).}$$

L^∞ control on the heat kernel

From here onwards, c, c_1, c_2, etc. denote some constants depending only on A and and which can be different from one line to another.

To conclude that section, we therefore have to find the L^∞ rate of decrease of H_α. By looking above at K_α and reminding that K_1 is controlled by cy/t, we have then to work on the second half of the sum, that is the amount

$$G(t,x,y) := K(t,x+y)\left(1+4A\sqrt{t}\,\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right)\right)$$

to know the L^∞ rate of decrease of H_α. Notice we have $\Gamma(z) = \frac{\sqrt{\pi}}{2}(\mathrm{Erf}(z)-1)$, and

$$\frac{\Gamma}{\gamma}(z) = -\frac{1}{2}\left(\frac{1}{z}-\frac{1}{2z^3}+o\left(\frac{1}{z^3}\right)\right) \qquad \text{as } z\to\infty.$$

Whence, in particular, if z_0 is taken large enough, we get for all $z \geq z_0$,

$$-\frac{1}{2z}+\frac{1}{8z^3} \leq \frac{\Gamma}{\gamma}(z) \leq -\frac{1}{2z}+\frac{3}{8z^3}.$$

Hence there exists $t_0 > 0$ also large enough such that for all $t \geq t_0$, all $x \geq 0$ and all $y \geq 0$,

$$\begin{aligned}\frac{\Gamma}{\gamma}\left(\frac{y+2tA+x}{2\sqrt{t}}\right) &\leq -\frac{1}{2}\frac{2\sqrt{t}}{2tA+x+y}+\frac{3}{8}\left(\frac{2\sqrt{t}}{2tA+x+y}\right)^3\\ &\leq -\frac{\sqrt{t}}{2tA+x+y}+\frac{3}{8(t)^{3/2}A^3}.\end{aligned}$$

Using that control on K_α, one obtains

$$\begin{aligned}K_\alpha(t,x,y) &= K_1(t,x,y)+2G(t,x,y)\\ &\leq \frac{cy}{t}+2K(t,x+y)\left(1+2A\sqrt{t}\left(-\frac{\sqrt{t}}{2tA+x+y}+\frac{3}{8(t)^{3/2}A^3}\right)\right)\\ &\leq \frac{cy}{t}+2K(t,x+y)\left(1-\frac{2tA}{2tA+x+y}+\frac{3}{4tA^2}\right)\\ &\leq \frac{cy}{t}+2K(t,x+y)\left(\frac{x+y}{2tA+x+y}+\frac{3}{4tA^2}\right)\\ &\leq \frac{cy}{t}+\frac{1}{\sqrt{\pi t}}\frac{x+y}{2tA+x+y}e^{-\frac{(x+y)^2}{4t}}+\frac{3}{4\sqrt{\pi}\,(t)^{3/2}A^2}\\ &= \frac{cy}{t}+\frac{c_1}{\sqrt{t}}\underbrace{\frac{x+y}{2tA+x+y}e^{-\frac{(x+y)^2}{4t}}}_{\text{Call that } (\spadesuit).}+\frac{c_2}{t^{3/2}}.\end{aligned}$$

There are two possible cases depending on the control which we shall obtain on $(\spadesuit)$:

- either $(\spadesuit) \leq c/t^\ell$ with $0 \leq \ell < 1/2$, and then K_α would be controlled by $c/t^{\frac{1}{2}+\ell}$,
- or $(\spadesuit) \leq c/t^\ell$ with $\ell \geq 1/2$, and then K_α would be controlled by $(cy+c_1)/t$.

The sequence consists then in controlling (♠). Let us pose

$$m(t) := \sup_{z \geq 0} \underbrace{\left(\frac{z}{2tA + z} e^{-\frac{z}{4t}} \right)}_{\text{and call that } \psi(z).}$$

One assess the first derivative of ψ:

$$\psi'(z) = -\frac{e^{-\frac{z^2}{4t}} \overbrace{\left(z^3 + 2tAz^2 - 4(t)^2 A \right)}^{\text{call that } \varphi(z)}}{2t(2tA + z)^2}.$$

We easily see that ψ' and φ have opposite signs. One has $\varphi'(z) = z(3z + 4tA)$ which remains non-negative when z is non-negative. Hence there exists some $z_* = z_*(t)$ such that $m(t) = \psi(z_*)$ as it can be seen from figure (*F40*). Remark that z_* is the unique real root of the polynomial $\varphi(z)$ and by seeing that, for all positive t,

$$\varphi\left(\sqrt{2t}\right) = (2t)^{3/2} > 0,$$

we actually have

$$m(t) = \max_{0 \leq z \leq \sqrt{2t}} \left(\frac{z}{2tA + z} e^{-\frac{z}{4t}} \right).$$

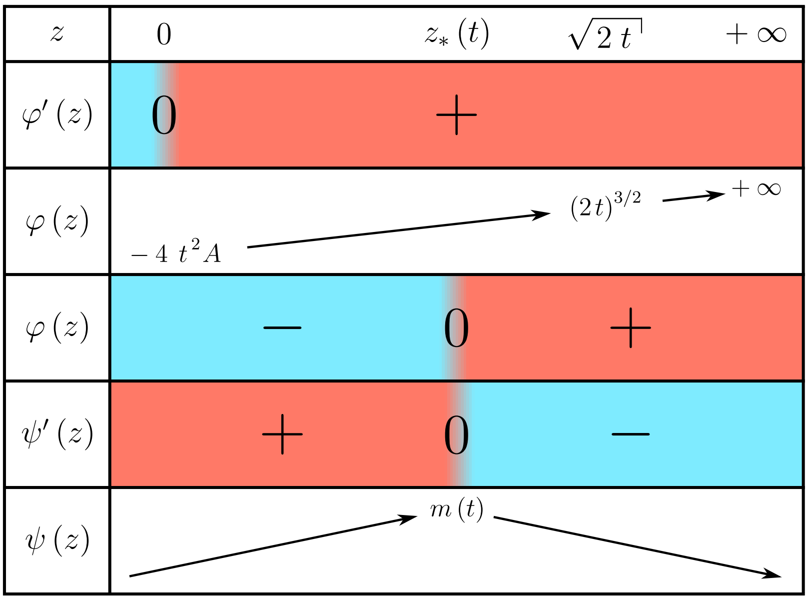


Figure F40 – *Study of the ψ function: $m(t)$ is actually a maximum achieved in z_*.*

Consequently, we can now control the amount $m(t)$ like this:

$$\begin{aligned} m(t) &= \max_{0 \leq z \leq \sqrt{2t}} \left(\frac{z}{2tA + z} e^{-\frac{z}{4t}} \right) \\ &\leq \max_{0 \leq z \leq \sqrt{2t}} \left(\frac{z}{2tA + z} \right) \end{aligned}$$

see that $\partial_z\left(\frac{z}{2tA+z}\right)=\frac{2tA}{(2tA+z)^2}>0$, whence

$$\begin{aligned} m(t) &\leq \frac{\sqrt{2t}}{2tA+2t} \\ &\leq \frac{\sqrt{2t}}{2tA} \\ &= \frac{1}{\sqrt{2t}\,A} \\ &= \frac{c}{\sqrt{t}}. \end{aligned}$$

Therefore (♠) is controlled by $\frac{c}{\sqrt{t}}$ so we get

$$K_\alpha(t,x,y)\leq \frac{cy}{t}+\frac{c_1}{t}+\frac{c_2}{t^{3/2}}$$

then, for t large enough,

$$K_\alpha(t,x,y)\leq \frac{cy}{t}+\frac{c_1}{t}+\frac{c_2}{t}$$

whence

$$\boxed{\left\|K_\alpha(t,\bullet,y)\right\|_{L^\infty}\leq \frac{cy+c_1}{t}.}$$

III.2 Heat kernel in the half-space $\mathbb{R}_+^N$

That section consists in a generalisation of the previous one in the N-dimensional case ($N\geq 1$). In the sequence, the set $\mathbb{R}_+^N$ denotes the upper half space $\mathbb{R}^{N-1}\times\mathbb{R}_+^*$ and a generic point $x\in\mathbb{R}^N$ shall be written

$$x=(X,x_N)\in\mathbb{R}^{N-1}\times\mathbb{R}.$$

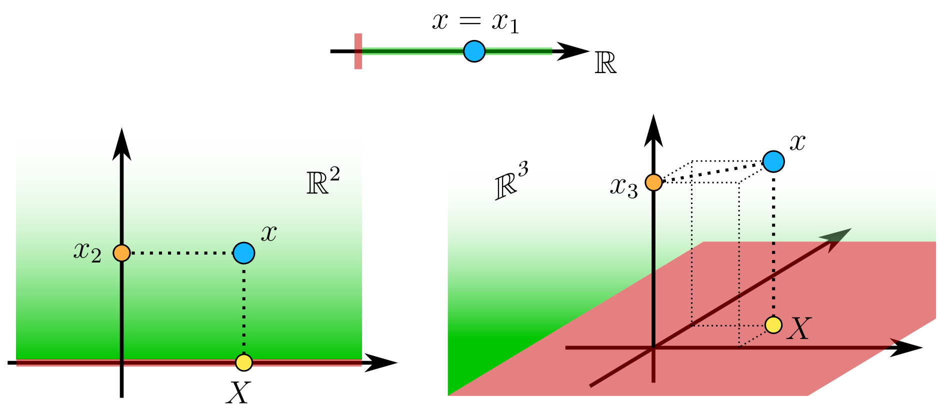


Figure F41 – *Representation of the generic point $x=$ ● $=$ (●, ●) $=(X,x_N)$ in the half spaces $\mathbb{R}_+^N$ for $N\in\{1,2,3\}$.*

One considers for non-negative valued $u_0 \in L^1\left(\mathbb{R}_+^N\right) \cap L^\infty\left(\mathbb{R}_+^N\right)$ the heat equation on the half space $\mathbb{R}_+^N$ provided with initial datum and Robin boundary conditions:

$$\left\{\begin{array}{ll} \partial_t u = \Delta u & (t,x) \in (0;\infty) \times \mathbb{R}_+^N \\ \alpha u\,(t,X,0) - (1-\alpha)\,\partial_{x_N} u\,(t,X,0) = 0 & (t,X) \in (0;\infty) \times \mathbb{R}^{N-1} \\ u\,(0,x) = u_0\,(x) & x \in \mathbb{R}_+^N \end{array}\right. \qquad \textbf{(III.3)}$$

where α is a real number in $[0;1]$. We are now going to do the same reasoning as which we have made in the previous section, that why a bit less details shall be given. We are thereby aiming to reach the α-Robin-$\mathbb{R}_+^N$-heat kernel $K_\alpha = K_\alpha\,(t,x,y)$ which gives the solution of (III.3) under the shape

$$u\,(t,x) = \int_{\mathbb{R}_+^N} K_\alpha\,(t,x,y)\,u_0\,(y)\,\mathrm{d}y.$$

Before starting we introduce the new notation $A \lesssim B$ which means that there is a positive constant $c = c\,(N,d,\alpha)$ such that $A \leq cB$ and we recall that the $\mathbb{R}^N$-heat kernel in the whole space is given by

$$K\,(t,x) \overset{\text{def}}{=} \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right),$$

or even, express with the notations given in the beginning of that section,

$$K\,(t,X,x_n) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|X|^2 + x_N^2}{4t}\right).$$

III.2.1 Neumann boundary conditions

Take here $\alpha = 0$ to get some Neumann boundary conditions in (III.3).

Proposition 42 (Heat equation in $\mathbb{R}_+^N$ with Neumann BC)

$\boxed{1}$ The 0-Robin-$\mathbb{R}_+^N$-heat kernel is given by

$$K_0\,(t,x,y) = K\,(t,X-Y,x_N-y_N) + K\,(t,X-Y,x_N+y_N)$$

$\boxed{2}$ Furthermore, one has the following L^∞ control on $K_0\,(t,\bullet,y)$:

$$\|K_0\,(t,\bullet,y)\|_{L^\infty\left(\mathbb{R}_+^N\right)} \lesssim \frac{1}{t^{N/2}}.$$

Proof (Proposition 42) $\boxed{1}$

Let $\widetilde{u_0}$ denote the even extension of u_0 with respect to the variable x_N beyond the hyperplane $\{x_N = 0\}$, namely

$$\widetilde{u_0}\,(X,x_N) := \left\{\begin{array}{ll} u_0\,(X,x_N) & \text{if } x_N > 0 \\ u_0\,(X,-x_N) & \text{if } x_N < 0. \end{array}\right.$$

We set $\widetilde{u} = \widetilde{u}(t, X, x_N)$ the solution of $\partial_t \widetilde{u} = \Delta \widetilde{u}$ in the whole $\mathbb{R}^N$ starting from the initial datum $\widetilde{u_0}$; take then for u the restriction of $\widetilde{u}$ to the upper half space, *i.e.* $u = \widetilde{u}_{|x_N>0}$. We assert that u is the solution of (III.3) with $\alpha = 0$. Indeed, we just have to check the boundary condition:

$$\begin{aligned}\widetilde{u}(t, X, -x_N) &= \int_{\mathbb{R}^N} K(t, X-Y, -x_N - y_N)\, \widetilde{u_0}(Y, y_N)\, \mathrm{d}Y \mathrm{d}y_N \\ &= \int_{\mathbb{R}^N} K(t, X-Y, -x_N + z_N)\, \widetilde{u_0}(Y, -z_N)\, \mathrm{d}Y \mathrm{d}z_N\end{aligned}$$

remind that K and $\widetilde{u_0}$ are even with respect to their last variable,

$$\begin{aligned}&= \int_{\mathbb{R}^N} K(t, X-Y, x_N - z_N)\, \widetilde{u_0}(Y, z_N)\, \mathrm{d}Y \mathrm{d}z_N \\ &= \widetilde{u}(t, X, x_N).\end{aligned}$$

Hence, $\widetilde{u}$ is even with respect to x_N that's why the boundary condition $\partial_{x_N} u(t, X, 0) = 0$ is verified. We assess now the fundamental solution K_0. We have, for all $t > 0$ and all $x \in \mathbb{R}^N_+$,

$$\begin{aligned}u(t, x) &= \int_{\mathbb{R}^N} K(t, x-y)\, \widetilde{u_0}(y)\, \mathrm{d}y \\ &= \int_{\mathbb{R}^N_+} K(t, X-Y, x_N - y_N)\, u_0(Y, y_N)\, \mathrm{d}Y \mathrm{d}y_N \\ &\qquad + \int_{\mathbb{R}^N_-} K(t, X-Y, x_N - y_N)\, \widetilde{u_0}(Y, y_N)\, \mathrm{d}Y \mathrm{d}y_N \\ &= \int_{\mathbb{R}^N_+} K(t, X-Y, x_N - y_N)\, u_0(Y, y_N)\, \mathrm{d}Y \mathrm{d}y_N \\ &\qquad + \int_{\mathbb{R}^N_+} K(t, X-Y, x_N + z_N)\, \widetilde{u_0}(Y, -z_N)\, \mathrm{d}Y \mathrm{d}z_N\end{aligned}$$

remind that $\widetilde{u_0}$ is even with respect to its last variable,

$$\begin{aligned}&= \int_{\mathbb{R}^N_+} K(t, X-Y, x_N - y_N)\, u_0(Y, y_N)\, \mathrm{d}Y \mathrm{d}y_N \\ &\qquad + \int_{\mathbb{R}^N_+} K(t, X-Y, x_N + z_N)\, u_0(Y, z_N)\, \mathrm{d}Y \mathrm{d}z_N \\ &= \int_{\mathbb{R}^N_+} \underbrace{\left[K(t, X-Y, x_N - y_N) + K(t, X-Y, x_N + y_N)\right]}_{\text{call that } K_0(t,x,y).\ \square} u_0(Y, y_N)\, \mathrm{d}Y \mathrm{d}y_N\end{aligned}$$

Proof (Proposition 42) $\boxed{2}$

That control easily follows from the expression of K_0. $\square$

III.2.2 Dirichlet boundary conditions

Take here $\alpha = 1$ to get some Dirichlet boundary conditions in (III.3).

Proposition 43 (Heat equation in $\mathbb{R}_+^N$ with Dirichlet BC)

$\boxed{1}$ The 1-Robin-$\mathbb{R}_+^N$-heat kernel is given by

$$K_1(t,x,y) = K(t, X-Y, x_N - y_N) - K(t, X-Y, x_N + y_N)$$

$\boxed{2}$ Furthermore, one has the following L^∞ control on $K_1(t,\bullet,y)$:

$$\|K_1(t,\bullet,y)\|_{L^\infty(\mathbb{R}_+^N)} \lesssim \frac{y_N}{t^{\frac{N+1}{2}}}.$$

Proof (Proposition 44) $\boxed{1}$

Let $\widetilde{u_0}$ denote the odd extension of u_0 with respect to the variable x_N beyond the hyperplane $\{x_N = 0\}$, namely

$$\widetilde{u_0}(X,x_N) := \begin{cases} u_0(X,x_N) & \text{if } x_N > 0 \\ -u_0(X,-x_N) & \text{if } x_N < 0. \end{cases}$$

We set $\tilde{u} = \tilde{u}(t,X,x_N)$ the solution of $\partial_t \tilde{u} = \Delta \tilde{u}$ in the whole $\mathbb{R}^N$ starting from the initial datum $\widetilde{u_0}$; take then for u the restriction of $\tilde{u}$ to the upper half space, *i.e.* $u = \tilde{u}_{|x_N>0}$. We assert that u is the solution of (III.3) with $\alpha = 1$. Indeed, we just have to check the boundary condition:

$$\begin{aligned}\tilde{u}(t,X,-x_N) &= \int_{\mathbb{R}^N} K(t,X-Y,-x_N-y_N)\,\widetilde{u_0}(Y,y_N)\,\mathrm{d}Y\mathrm{d}y_N \\ &= \int_{\mathbb{R}^N} K(t,X-Y,-x_N+z_N)\,\widetilde{u_0}(Y,-z_N)\,\mathrm{d}Y\mathrm{d}z_N\end{aligned}$$

remind that K is even and $\widetilde{u_0}$ is odd with respect to their last variable,

$$\begin{aligned}&= -\int_{\mathbb{R}^N} K(t,X-Y,x_N-z_N)\,\widetilde{u_0}(Y,z_N)\,\mathrm{d}Y\mathrm{d}z_N \\ &= -\tilde{u}(t,X,x_N).\end{aligned}$$

Hence, $\tilde{u}$ is odd with respect to x_N that's why the boundary condition $u(t,X,0) = 0$ is verified. We assess now the fundamental solution K_1. We have, for all $t > 0$ and all $x \in \mathbb{R}_+^N$,

$$\begin{aligned}u(t,x) &= \int_{\mathbb{R}^N} K(t,x-y)\,\widetilde{u_0}(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}_+^N} K(t,X-Y,x_N-y_N)\,u_0(Y,y_N)\,\mathrm{d}Y\mathrm{d}y_N \\ &\qquad + \int_{\mathbb{R}_-^N} K(t,X-Y,x_N-y_N)\,\widetilde{u_0}(Y,y_N)\,\mathrm{d}Y\mathrm{d}y_N\end{aligned}$$

$$= \int_{\mathbb{R}_+^N} K\left(t, X-Y, x_N - y_N\right) u_0\left(Y, y_N\right) \mathrm{d}Y \mathrm{d}y_N$$
$$+ \int_{\mathbb{R}_+^N} K\left(t, X-Y, x_N + z_N\right) \widetilde{u_0}\left(Y, -z_N\right) \mathrm{d}Y \mathrm{d}z_N$$

remind that $\widetilde{u_0}$ is odd with respect to its last variable,

$$= \int_{\mathbb{R}_+^N} K\left(t, X-Y, x_N - y_N\right) u_0\left(Y, y_N\right) \mathrm{d}Y \mathrm{d}y_N$$
$$- \int_{\mathbb{R}_+^N} K\left(t, X-Y, x_N + z_N\right) u_0\left(Y, z_N\right) \mathrm{d}Y \mathrm{d}z_N$$
$$= \int_{\mathbb{R}_+^N} \underbrace{\left[K\left(t, X-Y, x_N - y_N\right) - K\left(t, X-Y, x_N + y_N\right)\right]}_{\text{call that } K_1(t,x,y).\ \square} u_0\left(Y, y_N\right) \mathrm{d}Y \mathrm{d}y_N$$

Proof (Proposition 44) $\boxed{2}$
Take $\varphi\left(y_N\right) := \exp\left(-\frac{\left(x_N - y_N\right)^2}{4t}\right)$; in that way,

$$K_1\left(t, x, y\right) = \frac{1}{\left(4\pi t\right)^{N/2}} \exp\left(-\frac{\left|X-Y\right|^2}{4t}\right) \left(\varphi\left(y_N\right) - \varphi\left(-y_N\right)\right)$$
$$\leq \frac{1}{\left(4\pi t\right)^{N/2}} \left(\varphi\left(y_N\right) - \varphi\left(-y_N\right)\right).$$

Thanks to the Mean Value Theorem there exists some positive c such that

$$\left|\varphi\left(y_N\right) - \varphi\left(-y_N\right)\right| \leq \frac{c \cdot y_N}{\sqrt{t}}.$$

Whence for another positive c,

$$K_1\left(t, x, y\right) \leq \frac{c \cdot y_N}{t^{\frac{N+1}{2}}}. \quad \square$$

III.2.3 Robin boundary conditions

Take here $\alpha \in [0;1)$ to get some Robin boundary conditions in (III.3).

Proposition 44 (Heat equation in $\mathbb{R}_+^N$ with Robin BC)

$\boxed{1}$ The α-Robin-$\mathbb{R}_+^N$-heat kernel is given by

$$K_\alpha(t,x,y) = K_1(t,x,y) + 2K(t, X-Y, x_N+y_N) \times \left(1 + 2A\sqrt{t}\,\frac{\Gamma}{\gamma}\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right)\right).$$

$\boxed{2}$ Furthermore, one has the following L^∞ control on $K_\alpha(t,\bullet,y)$:

$$\|K_\alpha(t,\bullet,y)\|_{L^\infty\left(\mathbb{R}_+^N\right)} \lesssim \frac{1+y_N}{t^{\frac{N+1}{2}}}.$$

Proof (Proposition 42) $\boxed{1}$

Let α be in $[0;1)$ and state $A = \alpha/(1-\alpha)$. We moreover suppose in the sequence that the initial datum u_0 is derivable following the x_N variable.

Step 1: extend u_0 in the whole $\mathbb{R}^N$ to solve the half space problem.

We extend u_0 beyond the hyperplane $\{x_N = 0\}$ in a new initial datum $\widetilde{u_0} = \widetilde{u_0}(X, x_N)$ in such a way that the function

$$A\widetilde{u_0} - \partial_{x_N}\widetilde{u_0} \text{ is odd with respect to the } x_N \text{ variable.}$$

One therefore gets the following equivalent equalities:

$$A\widetilde{u_0}(X,x_N) - \partial_{x_N}\widetilde{u_0}(X,x_N) = -\left(A\widetilde{u_0}(X,-x_N) - \partial_{x_N}\widetilde{u_0}(X,-x_N)\right)$$

$$e^{-Ax_N}\left(\partial_{x_N}\widetilde{u_0}(X,x_N) - A\widetilde{u_0}(X,x_N)\right) = e^{-Ax_N}\left(A\widetilde{u_0}(X,-x_N) - \partial_{x_N}\widetilde{u_0}(X,-x_N)\right)$$

$$\partial_{x_N}\left(e^{-Ax_N}\widetilde{u_0}(X,x_N)\right) = e^{-Ax_N}\left(A\widetilde{u_0}(X,-x_N) - \partial_{x_N}\widetilde{u_0}(X,-x_N)\right)$$

$$\widetilde{u_0}(X,x_N) = Ce^{Ax_N} + e^{Ax_N}\int_0^{x_N} e^{-As}\left(A\widetilde{u_0}(X,-s) - \partial_{x_N}\widetilde{u_0}(X,-s)\right)\mathrm{d}s,$$

where $C = C(X)$ is constant with regards to the x_N variable and may be chosen equal to $u_0(\bullet,0)$ in order to $\widetilde{u_0}$ keeps its continuity when $x_N = 0$. Thereby we pose

$$\widetilde{u_0} := \begin{cases} u_0(X,x_N) & \text{if } x_N \geq 0, \\ u_0(X,0)\,e^{Ax_N} + e^{Ax_N}\int_0^{x_N} e^{-As}\left(Au_0(X,-s) - \partial_{x_N}u_0(X,-s)\right)\mathrm{d}s & \text{otherwise} \end{cases}$$

and we reach the solution of the heat equation starting from the initial datum $\widetilde{u_0}$:

$$\widetilde{u}(t,X,x_N) := \int_{\mathbb{R}^N} K(t, X-Y, x_N-y_N)\,\widetilde{u_0}(Y,y_N)\,\mathrm{d}Y\,\mathrm{d}y_N.$$

Claim. For all $t > 0$, $A\widetilde{u}(t, X, 0) - \partial_{x_N}\widetilde{u}(t, X, 0) = 0$.

Let's prove that by showing the oddness (with respect to x_N) of $A\widetilde{u} - \partial_{x_N}\widetilde{u}$:

$$
\begin{aligned}
A\widetilde{u}(t, X, -x_N) &- \partial_{x_N}\widetilde{u}(t, X, -x_N) \\
&= \int_{\mathbb{R}^N} K(t, Y, y_N)\, A\widetilde{u_0}(X - Y, -x_N - y_N)\, \mathrm{d}Y \mathrm{d}y_N \\
&\qquad - \int_{\mathbb{R}^N} K(t, Y, y_N)\, \partial_{x_N}\left(\widetilde{u_0}(X - Y, -x_N - y_N)\right) \mathrm{d}Y \mathrm{d}y_N \\
&= \int_{\mathbb{R}^N} K(t, Y, y_N) \Big[A\widetilde{u_0}(X - Y, -x_N - y_N) \\
&\qquad - \partial_{x_N}\left(\widetilde{u_0}(X - Y, -x_N - y_N)\right)\Big] \mathrm{d}Y \mathrm{d}y_N \\
&= \int_{\mathbb{R}^N} K(t, Y, y_N) \Big[\left(A\widetilde{u_0} - \partial_{x_N}\widetilde{u_0}\right)\left((X - Y, -x_N - y_N)\right)\Big] \mathrm{d}Y \mathrm{d}y_N
\end{aligned}
$$

take $z_N = -y_N$,

$$
= \int_{\mathbb{R}^N} K(t, Y, -z_N) \Big[\left(A\widetilde{u_0} - \partial_{x_N}\widetilde{u_0}\right)\left((X - Y, z_N - x_N)\right)\Big] \mathrm{d}Y \mathrm{d}z_N
$$

remind that $A\widetilde{u_0} - \partial_{x_N}\widetilde{u_0}$ is odd (with respect to x_N) and recall z_N by y_N,

$$
= -\int_{\mathbb{R}^N} K(t, Y, -y_N) \Big[\left(A\widetilde{u_0} - \partial_{x_N}\widetilde{u_0}\right)\left((X - Y, x_N - y_N)\right)\Big] \mathrm{d}Y \mathrm{d}y_N
$$

finally using that the $\mathbb{R}^N$-heat kernel K is even (with respect to x_N),

$$
\begin{aligned}
&= -\int_{\mathbb{R}^N} K(t, Y, y_N) \Big[\left(A\widetilde{u_0} - \partial_{x_N}\widetilde{u_0}\right)\left((X - Y, x_N - y_N)\right)\Big] \mathrm{d}Y \mathrm{d}y_N \\
&= -\Bigg(\int_{\mathbb{R}^N} K(t, Y, y_N)\, A\widetilde{u_0}(X - Y, -x_N - y_N)\, \mathrm{d}Y \mathrm{d}y_N \\
&\qquad - \int_{\mathbb{R}^N} K(t, Y, y_N)\, \partial_{x_N}\left(\widetilde{u_0}(X - Y, -x_N - y_N)\right) \mathrm{d}Y \mathrm{d}y_N\Bigg) \\
&= -\left(A\widetilde{u}(t, X, -x_N) - \partial_{x_N}\widetilde{u}(t, X, -x_N)\right).
\end{aligned}
$$

Whence the claimed result. $\square$ (claim)

One takes now for u the restriction of $\widetilde{u}$ to the upper half space $\{x_N \geq 0\}$, that is, $u = \widetilde{u}_{|x_N \geq 0}$, in that way u is the unique (thanks to the comparison principle) solution of problem (III.3).

Step 2: Find the α-Robin-$\mathbb{R}_+^N$-heat kernel K_α.

One has

$$\begin{aligned}
&u(t,X,x_N)\\
&= \int_{-\infty}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N-y_N)\,\widetilde{u_0}(Y,y_N)\,\mathrm{d}Y\,\mathrm{d}y_N\\
&= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N-y_N)\,u_0(Y,y_N)\,\mathrm{d}Y\,\mathrm{d}y_N\\
&\qquad + \int_{-\infty}^{0}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N-y_N)\,\widetilde{u_0}(Y,y_N)\,\mathrm{d}Y\,\mathrm{d}y_N\\
&= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N-y_N)\,u_0(Y,y_N)\,\mathrm{d}Y\,\mathrm{d}y_N\\
&\qquad + \underbrace{\int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\,\widetilde{u_0}(Y,-y_N)\,\mathrm{d}Y\,\mathrm{d}y_N}_{\text{call that integral } I(t,X,x_N).}
\end{aligned}$$

Let's work on the amount I:

$$\begin{aligned}
I(t,X,x_N) &= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\Big(u_0(Y,0)\,e^{-Ay_N}\\
&\qquad + e^{-Ay_N}\int_0^{-y_N} e^{-As}\left(Au_0(Y,-s)-\partial_{x_N}u_0(Y,-s)\right)\mathrm{d}s\Big)\mathrm{d}Y\,\mathrm{d}y_N\\
&= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\Big(u_0(Y,0)\,e^{-Ay_N}\\
&\qquad - e^{-Ay_N}\int_0^{y_N} e^{As}\left(Au_0(Y,s)-\partial_{x_N}u_0(Y,s)\right)\mathrm{d}s\Big)\mathrm{d}Y\,\mathrm{d}y_N\\
&= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\Big(u_0(Y,0)\,e^{-Ay_N}\\
&\qquad - Ae^{-Ay_N}\int_0^{y_N} e^{As}u_0(Y,s)\,\mathrm{d}s + e^{-Ay_N}\underbrace{\int_0^{y_N} e^{As}\partial_{x_N}u_0(Y,s)\,\mathrm{d}s}_{\text{IBP}}\Big)\mathrm{d}Y\,\mathrm{d}y_N\\
&= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\Big(u_0(Y,0)\,e^{-Ay_N}\\
&\qquad - Ae^{-Ay_N}\int_0^{y_N} e^{As}u_0(Y,s)\,\mathrm{d}s\\
&\qquad\quad + u_0(Y,y_N) - u_0(Y,0)\,e^{-Ay_N} - Ae^{-Ay_N}\int_0^{y_N} e^{As}u_0(Y,s)\,\mathrm{d}s\Big)\mathrm{d}Y\,\mathrm{d}y_N\\
&= \int_{0}^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\Big(u_0(Y,y_N)\\
&\qquad - 2Ae^{-Ay_N}\int_0^{y_N} e^{As}u_0(Y,s)\,\mathrm{d}s\Big)\mathrm{d}Y\,\mathrm{d}y_N.
\end{aligned}$$

We thus get

$$u(t,X,x_N) = \int_0^{+\infty}\int_{\mathbb{R}^{N-1}} \overbrace{\left(K(t,X-Y,x_N-y_N)+K(t,X-Y,x_N+y_N)\right)}^{\text{Recognize here } K_0(t,x,y)\ldots} u_0(Y,y_N)\,\mathrm{d}Y\,\mathrm{d}y_N$$
$$-2A\int_0^{+\infty}\int_{\mathbb{R}^{N-1}} K(t,X-Y,x_N+y_N)\,e^{-Ay_N}\left(\int_0^{y_N} e^{As}u_0(Y,s)\,\mathrm{d}s\right)\mathrm{d}Y\,\mathrm{d}y_N.$$

In the second term of the latter sum, one can check that

$$K(t,X-Y,x_N+y_N)\,e^{-Ay_N} = \frac{e^{-\frac{|X-Y|^2}{4t}}}{(4\pi t)^{N/2}}\cdot\exp\left(-\frac{(y_N+(2tA+x_N))^2-(4tAx_N+4d^2t^2A^2)}{4t}\right)$$

then by letting $\gamma(z) := e^{-z^2}$, we have

$$K(t,X-Y,x_N+y_N)\,e^{-Ay_N} = \frac{e^{-\frac{|X-Y|^2}{4t}}}{(4\pi t)^{N/2}}\cdot\exp\left(Ax_N+tA^2\right)\gamma\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right).$$

Therefore,

$$u(t,X,x_N) = \int_{\mathbb{R}_+^N} K_0(t,x,y)\,u_0(y)\,\mathrm{d}y - \frac{2Ae^{Ax_N+tA^2}}{(4\pi t)^{N/2}}$$
$$\times\int_{\mathbb{R}^{N-1}} e^{-\frac{|X-Y|^2}{4t}} \underbrace{\int_0^{+\infty}\left(\gamma\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right)\int_0^{y_N} e^{As}u_0(Y,s)\,\mathrm{d}s\right)\mathrm{d}y_N}_{\text{call that } I_1(t,X,x_N,Y)\text{, we shall do an IBP on it.}}\,\mathrm{d}Y$$

We pose $\Gamma(z_N) := -\int_{z_N}^{+\infty}\gamma(s)\,\mathrm{d}s$; in that way, Γ is a primitive of γ which is zero when z_N tends to $+\infty$ and by integrating by part, as specified just above,

$$I_1(t,X,x_N,Y) = -2\sqrt{t}\int_0^{+\infty}\Gamma\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right)e^{Ay_N}u_0(Y,y_N)\,\mathrm{d}y_N.$$

So u becomes

$$u(t,X,x_N) = \int_{\mathbb{R}_+^N} K_0(t,x,y)\,u_0(y)\,\mathrm{d}y + \frac{4A\sqrt{t}}{(4\pi t)^{N/2}}$$
$$\times\int_{\mathbb{R}^{N-1}} e^{-\frac{|X-Y|^2}{4t}} \underbrace{\int_0^{+\infty} e^{Ax_N+tA^2+Ay_N}\cdot\Gamma\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right)u_0(Y,y_N)\,\mathrm{d}y_N}_{\text{call that } I_2(t,y',x_N).}\,\mathrm{d}Y$$

Working on I_2:

$$\begin{aligned} I_2\left(t, y', x_N\right) = &\int_0^{+\infty} e^{Ax_N + tA^2 + Ay_N} \cdot \exp\left(-\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right)^2\right) \\ &\times \Gamma\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right) \cdot \exp\left(+\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right)^2\right) u_0\left(Y, y_N\right) \mathrm{d}y_N \\ = &\int_0^{+\infty} \exp\left(-\frac{\left(x_N + y_N\right)^2}{4t}\right) \cdot \frac{\Gamma}{\gamma}\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right) u_0\left(Y, y_N\right) \mathrm{d}y_N. \end{aligned}$$

We finally get

$$\begin{aligned} u\left(t, X, x_N\right) = \int_{\mathbb{R}_+^N} \Bigg(&K_0\left(t, x, y\right) + 4A\sqrt{t}\, K\left(t, X - Y, x_N + y_N\right) \\ &\times \frac{\Gamma}{\gamma}\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right)\Bigg) u_0\left(Y, y_N\right) \mathrm{d}Y \mathrm{d}y_N. \end{aligned}$$

consequently we have found the α-Robin-$\mathbb{R}_+^N$-heat kernel:

$$K_\alpha\left(t, x, y\right) = K_0\left(t, x, y\right) + 4A\sqrt{t}\, K\left(t, X - Y, x_N + y_N\right) \frac{\Gamma}{\gamma}\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right).$$

We take the liberty of changing a little bit its shape:

$$\begin{aligned} K_\alpha\left(t, x, y\right) &= K\left(t, X - Y, x_N - y_N\right) + K\left(t, X - Y, x_N + y_N\right) \\ &\quad + 4A\sqrt{t}\, K\left(t, X - Y, x_N + y_N\right) \frac{\Gamma}{\gamma}\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right) \\ &= K\left(t, X - Y, x_N - y_N\right) - K\left(t, X - Y, x_N + y_N\right) + 2K\left(t, X - Y, x_N + y_N\right) \\ &\quad + 4A\sqrt{t}\, K\left(t, X - Y, x_N + y_N\right) \frac{\Gamma}{\gamma}\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right). \end{aligned}$$

Thereby,

$$\boxed{K_\alpha\left(t, x, y\right) = K_1\left(t, x, y\right) + 2K\left(t, X - Y, x_N + y_N\right)\left(1 + 2A\sqrt{t}\, \frac{\Gamma}{\gamma}\left(\frac{y_N + 2tA + x_N}{2\sqrt{t}}\right)\right).}$$

Step 3: Show that K_α works also for non-regular $u_0 \in L^1\left(\mathbb{R}_+^N\right) \cap L^\infty\left(\mathbb{R}_+^N\right)$.

The derivability hypothesis we have done on the initial datum u_0 was necessary to find the α-Robin-$\mathbb{R}_+^N$-heat kernel but the normal derivative of u_0 is not involved in the final expression of u given by

$$u\left(t, x\right) = \int_{\mathbb{R}_+^N} K_\alpha\left(t, x, y\right) u_0\left(y\right) \mathrm{d}y.$$

This being due to an integration by part we have done above allowing K_α to hold implicitly the normal derivative instead of u_0. Therefore, the formula giving u by integrating K_α against u_0 does not require further hypothesis than $u_0 \in L^1\left(\mathbb{R}^N_+\right) \cap L^\infty\left(\mathbb{R}^N_+\right)$. That's why we would be happy if our α-Robin-$\mathbb{R}^N_+$-heat kernel does also work for such initial datum.

We first check that K_α verifies the Robin boundary condition on $\partial\mathbb{R}^N_+$: we call

$$\theta := \frac{y_N + 2tA + x_N}{2\sqrt{t}} \qquad \text{and} \qquad \theta_0 := \frac{y_N + 2tA}{2\sqrt{t}},$$

some easy but long computations show that

$$K_\alpha\left(t, (X,0), (Y, y_N)\right) = K\left(t, X-Y, y_N\right)\left(2 + 4A\sqrt{t}\right)\frac{\Gamma}{\gamma}\left(\theta_0\right),$$

$$\begin{aligned}\partial_{x_N} K_\alpha\left(t, (X,x_N), (Y, y_N)\right) = &-\frac{x_N - y_N}{2t} K\left(t, X-Y, x_N - y_N\right)\\ &-\frac{x_N + y_N}{2t} K\left(t, X-Y, x_N + y_N\right)\\ &+ K\left(t, X-Y, x_N + y_N\right)\left[2A - 4\sqrt{t}\,A^2\frac{\Gamma}{\gamma}\left(\theta\right)\right],\end{aligned}$$

$$\partial_{x_N} K_\alpha\left(t, (X,0), (Y, y_N)\right) = K\left(t, X-Y, y_N\right)\left[2A - 4\sqrt{t}\,A^2\frac{\Gamma}{\gamma}\left(\theta_0\right)\right].$$

Hence one sees that

$$AK_\alpha\left(t, (X,0), (Y, y_N)\right) - \partial_{x_N} K_\alpha\left(t, (X,0), (Y, y_N)\right) = 0$$

that is $K_\alpha$verifies Robin boundary conditions on $\partial\mathbb{R}^N_+$ for all positive time. Let now $u_0 \in L^1\left(\mathbb{R}^N_+\right) \cap L^\infty\left(\mathbb{R}^N_+\right)$ and pose

$$u\left(t,x\right) = \begin{cases} u_0\left(x\right) & \text{if } t = 0\\ \int_{\mathbb{R}^N_+} K_\alpha\left(t,x,y\right) u_0\left(y\right) \mathrm{d}y. & \text{if } t > 0.\end{cases}$$

It can be verified from one part that $\partial_t u = \Delta u$ for all $(t,x) \in \mathbb{R}^*_+ \times \mathbb{R}^N_+$, and from another part, we have

$$\begin{aligned}Au\left(t,(X,0)\right) &- \partial_{x_N} u\left(t,(X,0)\right)\\ &= \int_{\mathbb{R}^N_+} \left[AK_\alpha\left(t, (X,0), (Y, y_N)\right) - \partial_{x_N} K_\alpha\left(t, (X,0), (Y, y_N)\right)\right] u_0\left(y\right) \mathrm{d}y\\ &= 0.\end{aligned}$$

Therefore, the function is well the solution of problem (III.3) starting from the not-necessary-regular initial datum u_0. □

Proof (Proposition 42) $\boxed{2}$
From here onwards, c, c_1, c_2, etc. denote some constants depending only on N, α and d and which can be different from one line to another. By looking at K_α expression assessed in first part of the proof and reminding that K_1 is controlled by $cy_N/t^{\frac{N+1}{2}}$, we therefore have to work on the second half of K_α, that is

$$G(t,x,y) := K(t, X-Y, x_N+y_N)\left(1+2A\sqrt{t}\,\frac{\Gamma}{\gamma}\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right)\right)$$

to know the L^∞ rate of decrease of K_α. Notice we have $\Gamma(z_N) = \frac{\sqrt{\pi}}{2}\left(\mathrm{Erf}(z_N)-1\right)$, and

$$\frac{\Gamma}{\gamma}(z_N) = -\frac{1}{2}\left(\frac{1}{z_N} - \frac{1}{2z_N^3} + o\left(\frac{1}{z_N^3}\right)\right) \qquad \text{as } z_N \to \infty.$$

Whence, in particular, if z_0 is taken large enough, we get for all $z_N \geq z_0$,

$$-\frac{1}{2z_N} + \frac{1}{8z_N^3} \leq \frac{\Gamma}{\gamma}(z_N) \leq -\frac{1}{2z_N} + \frac{3}{8z_N^3}.$$

Hence there exists $t_0 > 0$ also large enough such that for all $t \geq t_0$, all $x_N \geq 0$, and all $y_N \geq 0$,

$$\begin{aligned}
\frac{\Gamma}{\gamma}\left(\frac{y_N+2tA+x_N}{2\sqrt{t}}\right) &\leq -\frac{1}{2}\frac{2\sqrt{t}}{y_N+2tA+x_N} + \frac{3}{8}\left(\frac{2\sqrt{t}}{y_N+2tA+x_N}\right)^3 \\
&\leq -\frac{\sqrt{t}}{y_N+2tA+x_N} + \frac{3}{8(t)^{3/2}A^3}.
\end{aligned}$$

Using that control on K_α, one obtains

$$\begin{aligned}
K_\alpha(t,x,y) &= K_1(t,x,y) + 2G(t,x,y) \\
&\leq \frac{cy_N}{t^{\frac{N+1}{2}}} + 2K(t, X-Y, x_N+y_N) \\
&\qquad \times \left(1+2A\sqrt{t}\left(-\frac{\sqrt{t}}{y_N+2tA+x_N} + \frac{3}{8(t)^{3/2}A^3}\right)\right) \\
&\leq \frac{cy_N}{t^{\frac{N+1}{2}}} + 2K(t, X-Y, x_N+y_N)\left(1 - \frac{2tA}{y_N+2tA+x_N} + \frac{3}{4tA^2}\right) \\
&\leq \frac{cy_N}{t^{\frac{N+1}{2}}} + 2K(t, X-Y, x_N+y_N)\left(\frac{x_N+y_N}{y_N+2tA+x_N} + \frac{3}{4tA^2}\right) \\
&\leq \frac{cy_N}{t^{\frac{N+1}{2}}} + \frac{2}{(4\pi t)^{N/2}}\frac{x_N+y_N}{y_N+2tA+x_N}e^{-\frac{|X-Y|^2+(x_N+y_N)^2}{4t}} + \frac{3}{2(4\pi)^{N/2}(t)^{1+\frac{N}{2}}A^2} \\
&= \frac{cy_N}{t^{\frac{N+1}{2}}} + \frac{c_1}{t^{N/2}}\frac{x_N+y_N}{y_N+2tA+x_N}e^{-\frac{(x_N+y_N)^2}{4t}}e^{-\frac{|X-Y|^2}{4t}} + \frac{c_2}{t^{\frac{N+2}{2}}} \\
&\leq \frac{cy_N}{t^{\frac{N+1}{2}}} + \frac{c_1}{t^{N/2}}\underbrace{\frac{x_N+y_N}{y_N+2tA+x_N}e^{-\frac{(x_N+y_N)^2}{4t}}}_{\text{Call that } (\spadesuit).} + \frac{c_2}{t^{\frac{N+2}{2}}}.
\end{aligned}$$

The sequence consists then in controlling (♠). To do this, one refers the reader to the reasoning done at pages 93-94 where same control has been done. Finally one obtains that

$$(\spadesuit) \leq \frac{c}{\sqrt{t}}$$

so we get

$$K_\alpha(t,x,y) \leq \frac{cy_N}{t^{\frac{N+1}{2}}} + \frac{c_1}{t^{\frac{N+1}{2}}} + \frac{c_2}{t^{\frac{N+2}{2}}},$$

then for t large enough,

$$K_\alpha(t,x,y) \leq \frac{cy_N}{t^{\frac{N+1}{2}}} + \frac{c_1}{t^{\frac{N+1}{2}}} + \frac{c_2}{t^{\frac{N+1}{2}}},$$

whence

$$\boxed{\|K_\alpha(t,\bullet,y)\|_{L^\infty\left(\mathbb{R}_+^N\right)} \leq \frac{cy_N + c_1}{t^{\frac{N+1}{2}}}.} \quad \square$$

Remark. Notice one may show by an easy computation that the way we have extended u_0 in $\widetilde{u_0}$ is the only one which allows to obtain the Robin condition

$$A\widetilde{u}(t,X,0) - \partial_{x_N}\widetilde{u}(t,X,0)$$

for all $t > 0$. Therefore no simpler extension of u_0 can be found.

Scripts for numerical implementation

We have placed here the code we use to obtain the numerical results we have presented along this report. The programs used are Scilab and FreeFem++ which are both free downloadable. It is specified, at the beginning of each script which software is used. You shall observe sometimes the symbols $****$ in comment, they indicate that the value they follow may be changed without compromising the integrity of the script.

1. R-D equation in $\mathbb{R}^2$ ((*F12*) page 26) and ((*F19*) page 36)

```
//(FreeFem++)
//The following code allows to see some R-D equations in a
    rectangular domain of R2 for the reaction f(u)=u^(1+p)(1-u)

real T = 25; //Final time****
real dt = 0.1; //Time step****

int J=2; //Number of adjustments of the mesh (>0)

int n = 100; //Number of subdivisions****

real H = 3; //Height****
real W = 3; //Width****

real d = 0.01;  //Diffusion intensity****

//FUJITA'S EXPONENT
// p Fujita = pF = 2/N = 2/2 = 1
// p<pF => systematic HTE
// p>pF => non-systematic HTE
// p=0  => logistic
real p = 0; //Allee Effect intensity****

//COMPACTLY SUPPORTED INITIAL DATUM (Indicator function of a disc)
real x0 = 0; //x-coordinate of the disc****
real y0 = H/2; //y-coordinate of the disc****
real radius = 0.2; //Radius of the disc****
real u0max = 0.7; //Maximum of the initial datum (<1)
func InitialDatum = u0max*0.5*(1+sign(radius - sqrt((x-x0)^2+(y-y0
    )^2))); //Building of the initial datum

//COMPUTE TO GET A REGULAR MESH
real ratio = H/W;
int m = n*W/H;
```

```

//BUILDING OF THE RECTANGULAR MESH
border bottom(t = 0,1) {label = 1 ; x = -W/2 + t*W ; y = 0 ;};
border right(t = 0,1) {label = 1 ; x = W/2 ; y = 0 + t*H ;};
border top(t = 0,1) {label = 1 ; x = W/2 - t*W ; y = H ;};
border left(t = 0,1) {label = 1 ; x = -W/2 ; y = H - t*H ;};
mesh Th=buildmesh(bottom(m)+right(n)+top(m)+left(n));

//plot(Th, wait=false); //Display the mesh

//DEFINITION OF FUNCTIONAL SPACES AND FUNCTIONS
fespace Vh(Th,P1);
Vh u0=InitialDatum;
Vh u=u0, v, uold;

//PLOT THE INITIAL DATUM
real[int] colorhsv=[ // To plot in black & white
 0, 0 , 0,   // min is in black (note min is here zero)
 0, 0 , u0max  // u0max is in grey (1 is in white)
 ];

cout << endl<< endl <<"Norm L1 of u0 = " << int2d(Th)(u0) << endl << endl<< endl; //Display the L1-norm of u0

plot(u, nbiso=255, value = 0, dim = 2, fill = 1, wait=1, hsv=colorhsv, cmm="Press ENTER");

//DEFINE THE WEAK PROBLEM
problem RD(u, v, solver=UMFPACK) //v is the test function
 = int2d(Th)(u*v/dt) - int2d(Th)(uold*v/dt) //Estimate the time derivative

 + int2d(Th)(d* (dx(u)*dx(v)+dy(u)*dy(v))) //Diffusion

 - int2d(Th)(uold^(1+p)*(1-uold)*v) //Reaction

    //+ on(1, u=0)  //Dirichlet on the frontier boundaries (if commented : Neumann)
;

{
ofstream savemin("min.dat"); //To save min and max of the solution for each time
ofstream savemax("max.dat");

//SOLVING THE PROBLEM AND SAVE MIN AND MAX
for(real t = 0; t <= T; t += dt){
 uold = u; //ie. u^{n-1} = u^n

 for (int j=1;j<J+1;j++)
 {
 RD;  //Solve the problem RD
 Th=adaptmesh(Th,u);  //Adapt the mesh to the function u
```

```
 //plot(Th,wait=1);  //To see how adaptmesh works...
 }

    uold=abs(uold); //To get rid of potential negative values due to approximation
    u=abs(u);  //Idem

 real umax=uold[].max;  //Get the min and the max of the solution
 real umin=uold[].min;

 cout << "min = " << umin << endl;
 cout << "max = " << umax << endl;

    //real uminRounded = round(abs(umin)*100)/100; //The rounded min and max
    //real umaxRounded = round(abs(umax)*100)/100;

 real[int] colorhsv=[ // To plot in black & white
  0, 0 , umin, // min is in black
  0, 0 , umax  // max is in white
 ];

    plot(u, nbiso=255, value = 0, cmm="Time="+t+"          min="+umin+"          max="+umax, dim = 2, fill = 1, wait=0, hsv=colorhsv); //Plot the solution

    savemin << t << " " << umin << endl; //save the solution's min
    savemax << t << " " << umax << endl; //save the solution's max
}//END SOLVING PROBLEM

}//END OFSTREAM
```

2. R-D equation on the Field-Road space $\mathbb{R}^2_+$ ((*F27*) page 55)

```
//(FreeFem++)
//The following code allows to see some R-D equations on the Road-Field space R^{2}_{+} for the reaction f(u)=u^(1+p)(1-u)

real T = 25; //Final time****
real dt = 0.1; //Time step****

int n = 20; //Number of subdivisions****

real H = 3;  //Height****
real W = 10; //Width****

//PARAMETERS OF THE ROAD-FIELD SPACE
real d = 0.01;  //Diffusion intensity in the Field****
real D = 1;  //Diffusion intensity on the Road****
real mu = 4;  //Migratory equilibrium between Road and Field****

//FUJITA'S EXPONENT
// p Fujita = pF = 2/N = 2/2 = 1
// p<pF => systematic HTE
// p>pF => non-systematic HTE
// p=0  => logistic
real p = 0; //Allee Effect intensity****

//COMPACTLY SUPPORTED INITIAL DATUM ON FIELD (Indicator function of a disc)
real x0 = 0; //x-coordinate of the disc****
real y0 = H/16; //y-coordinate of the disc****
real radius = 0.2; //Radius of the disc****
real v0max = 0.7; //Maximum of the initial datum (<1)
func InitialDatum = v0max*0.5*(1+sign(radius - sqrt((x-x0)^2+(y-y0)^2))); //Building of the initial datum

//COMPUTE TO GET A REGULAR MESH
real ratio = H/W;
int m = n*W/H;

int NoRoad = 1; //label of truncation zones of the space (put Neumann on that)
int Road = 2; //label of the interface of exchange (link between Field and Road)

//BUILDING OF THE RECTANGULAR MESH
border bottom(t = 0,1) {label = 2 ; x = -W/2 + t*W ; y = 0 ;};
border right(t = 0,1) {label = 1 ; x = W/2 ; y = 0 + t*H ;};
border top(t = 0,1) {label = 1 ; x = W/2 - t*W ; y = H ;};
border left(t = 0,1) {label = 1 ; x = -W/2 ; y = H - t*H ;};
mesh Th=buildmesh(bottom(m)+right(n)+top(m)+left(n));

```

```
//plot(Th, wait=false); //Display the mesh

//DEFINITION OF FUNCTIONAL SPACES AND FUNCTIONS
fespace Vh(Th,P1);
Vh v0=InitialDatum;
Vh v=v0, w1, vold; //Field's functions (w1 is test function)
Vh u=0, w2, uold ; //Road's functions (w2 is test function)

//PLOT THE INITIAL DATUM
real[int] colorhsv=[ // To plot in black(0) & white(1)
 0, 0 , 0,   // min is in black (note min is here zero)
 0, 0 , v0max  // u0max is in grey (1 is in white)
 ];
plot(v, nbiso=255, value = 0, dim = 2, fill = 1, wait=1, hsv=
   colorhsv, cmm="Press ENTER");

//DEFINE THE WEAK PROBLEMS (FIELD & ROAD)
problem field(v, w1) //w1 is the test function
 = int2d(Th)(v*w1/dt)-int2d(Th)(vold*w1/dt) //Estimate the time
   derivative

 + int2d(Th)(d* (dx(v)*dx(w1)+dy(v)*dy(w1))) //Diffusion

 + int1d(Th, 2)(v*w1) //Migration Field>>Road
 -int1d(Th, 2)(mu*u*w1) //Migration Road>>Field

 - int2d(Th)(vold^(1+p)*(1-vold)*w1) //Reaction

 //+ on(1, v=0)  //Dirichlet on the frontier boundaries (if
   commented : Neumann)
;

//Note the u is considered as a function whom the domain is the
   same as v but it does not matter because we only consider u at
   the bottom frontier...
problem road(u, w2)  //w2 is the test function
    = int2d(Th)(u*w2/dt)-int2d(Th)(uold*w2/dt) //Estimate the time
    derivative

    + int2d(Th)(D* (dx(u)*dx(w2))) //Diffusion

 + int2d(Th)(mu*u*w2) //Migration Road>>Field
 - int2d(Th)(v*w2)  //Migration Field>>Road

 //+ on(1, u=0  //Dirichlet on the frontier boundaries (if
   commented : Neumann)
;

{
ofstream usavemin("umin.dat");
ofstream usavemax("umax.dat");
ofstream vsavemin("vmin.dat");
ofstream vsavemax("vmax.dat");
```

```

//SOLVING THE PROBLEMS AND SAVE MIN AND MAX
for(real t = 0; t <= T+dt; t += dt){
 uold = u; //ie. u^{n-1} = u^n
 road;  //Solve road problem

 vold = v; //ie. u^{n-1} = u^n
 field;  //Solve field problem

 uold=abs(uold); //To get rid of potential negative values due to approximation
 vold=abs(vold); //Idem
 u=abs(u);  //Idem
 v=abs(v);  //Idem

 real vmax=vold[].max;  //Get the min and the max of v
    real vmin=vold[].min;

    real vminRounded = round(abs(vmin)*100)/100; //The rounded min and max of v
    real vmaxRounded = round(abs(vmax)*100)/100;

 real[int] colorhsv=[ // To plot in black(0) & white(1)
  0, 0 , vmin,
  0, 0 , vmax
 ];

    //GET VALUES ON THE ROAD (we need to extract the values of u on the bottom frontier)
 varf On2(u,v) = on(2,u=1);
 real[int] on2=On2(0,Vh,tgv=1);
 int[int] indices2(on2.sum);
 for(int i=0,j=0;i<Vh.ndof;++i) if(on2[i]){
  indices2[j] = i; ++j;
 }
 Vh uu=u;
 real[int] uon2(indices2.n);
 for(int i=0;i<indices2.n;++i){
  uon2[i] = uu[][indices2[i]];
 }

 real umax=uon2.max;  //Get the min and the max of u
 real umin=uon2.min;

 real uminRounded = round(abs(umin)*100)/100; //The rounded min and max of u
 real umaxRounded = round(abs(umax)*100)/100;

 plot(v, nbiso=255, cmm="(Plot : FIELD)     (Time = "+t+" on "+T+")       (Min = "+uminRounded+" | "+vminRounded+")      (Max = "+umaxRounded+"  |  "+vmaxRounded+")       (Road diffusion : D = "+D+")      (Field diffusion : d = "+d+")      (Migration equilibrium : mu = "+mu+")", value = 0, dim = 2, fill = 1, wait
```

```
=0, hsv=colorhsv);

 usavemin << t << " " << umin << endl;//save the solution's min
 usavemax << t << " " << umax << endl;//save the solution's max

 vsavemin << t << " " << vmin << endl;//save the solution's min
 vsavemax << t << " " << vmax << endl;//save the solution's max
}//END SOLVING PROBLEMS

}//END OFSTREAM
```

3. Spreading speed for large Road diffusion ((*F37*) page 80)

```
//(Scilab)
//The following code allows to assess with epsilon-accuracy the value of the asymptotic speed C*=C*(\mu, d, D)=C_ast according to the parameters \mu, d and D.

clear()

//ACCURACY PARAMETERS
epsilon=1e-3; //Closeness to the right C*
n=1000;   //Thinness of the discretization to check the positivity condition on phi

//DEFINITION OF FUNCTION \varphi
function y=phi(c,b)
 y=(c-sqrt(c*c-CKPP*CKPP - 4*d*d*b*b))/(2*d) - (c+sqrt(c*c+(4*mu*d*D*b)/(1+d*b)))/(2*D);
endfunction

//ESTIMATE FOR C*
function result=c_ast(mu,d,D)
 CKPP=2*sqrt(d*dx_f_zero);

 //Bounds fo finding C*
 c0=CKPP;//Value of c for which we are sure there exists no solution
 c1=dx_f_zero*sqrt(D)+1;//Value of c for which we are sure there exists two solutions

 phi_values=zeros(1,n);  //Initialisation for saving the values of phi

 //DICHOTOMOUS LOOP
 while (c1-c0>epsilon)   //Repeat the loop while accuracy is not sufficient...
  c_bar=(c0+c1)/2;

  //Discretisation of the place were \beta sits for checking positivity of phi
  space=linspace(0,sqrt(c_bar*c_bar-CKPP*CKPP)/(2*d),n);

  //Store the values of phi
  for i=1:n
    phi_values(i)=phi(c_bar,space(i));
  end

  //Check whether phi is positive
  if min(real(phi_values))>0 then
   c0=c_bar;
  else
```

```
   c1=c_bar;
  end

 end
 result=c_bar;   //Return the result of the function
endfunction

//PARAMETERS
mu=1;   //****
d=1; //****
D=10; //****
dx_f_zero=1;  //f'(0)=1
c_ast(mu,d,D)//This is a test; with \mu=d=f'(0)=1 and D=10, one finds on Geogebra C*~=3.2

//////////////////////////////////////
//The following concern the behaviour of C* as D becomes large; mu and d being fixed =1. One assess C*(1,1,D) for many large values of D then we plot (D, C*(1,1,D)) in a LOG/LOG scale; we expect then the point to be aligned on the graph following the direction of y=x/2.

mu=1;
d=1;
dx_f_zero=1;

D_values=%e^(linspace(1,37,100)); //Values tested of D (adapted to the LOG scale)
C_ast_values=zeros(D_values);    //Initialisation of the store of values of C*

for i=1:length(D_values)
  C_ast_values(i)=c_ast(mu,d,D_values(i));
end

LOG_D_values=log(D_values);
LOG_C_ast_values=log(C_ast_values);

//PLOTING IN THE LOG/LOG scale
clf()
plot(LOG_D_values,LOG_C_ast_values,'o') //plot the points
plot(LOG_D_values,LOG_C_ast_values,) //interpolate the points
plot([1 37],[2 20]) //plot the line y=x/2 + 3/2 to compare the slopes

correlation=corr(LOG_D_values,LOG_C_ast_values,1)/sqrt(corr(LOG_D_values,LOG_D_values,1)*corr(LOG_C_ast_values,LOG_C_ast_values,1)); //assess the linear correlation coefficient

[a,b,s]=reglin(LOG_D_values,LOG_C_ast_values); //assess the equation's coefficients of the line

```

```
disp('As D becomes large, C* evolves as '+string(exp(b))+'(D)^('+
    string(a)+'). The linear correlation coefficient equals '+
    string(correlation))
```

Toolbox

This part is dedicated to alleviate the content of this report. We announce here some general results whom enunciation at the moment we use them may be unwelcome.

1. Fourier transform

For a function $f \in L^1(\mathbb{R})$, one calls *Fourier transform of* f the function defined for all $\xi \in \mathbb{R}$ by

$$\mathscr{F}[f](\xi) = \hat{f}(\xi) := \int_{\mathbb{R}} \exp(-ix\xi) f(x) \,\mathrm{d}x,$$

and one calls *inverse Fourier transform of* f the function defined for all $x \in \mathbb{R}$ by

$$\mathscr{F}^{-1}[f](\xi) = \check{f}(x) := \int_{\mathbb{R}} \exp(+i\xi x) f(\xi) \,\mathrm{d}\xi.$$

One states now a few properties for the Fourier transform:

$\boxed{1}$ $\hat{f}$ is bounded and $\left|\hat{f}\right| \leq \|f\|_{L^1}$.

$\boxed{2}$ $\widehat{f * g} = \hat{f} \cdot \hat{g}$, where $\cdot$ denotes the pointwise product.

$\boxed{3}$ f may be rebuilt from $\hat{f}$ thanks to the following inversion formula:

$$f = \frac{1}{2\pi} \mathscr{F}^{-1}[\mathscr{F}[f]] = \frac{1}{2\pi} \check{\hat{f}}.$$

$\boxed{4}$ Gaussian functions remain Gaussian through the Fourier transform: for $a > 0$,

$$\mathscr{F}\left[\exp\left(-ax^2\right)\right] = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{1}{4a}\xi^2\right).$$

$\boxed{5}$ Assume that $f \in \mathcal{C}_c^2(\mathbb{R})$ or $f \in \mathcal{S}(\mathbb{R})$, then we have

$$\mathscr{F}[f'](\xi) = i\xi \hat{f}(\xi) \qquad \text{and} \qquad \mathscr{F}[f''](\xi) = -\xi^2 \hat{f}(\xi).$$

2. Eigenvalues of the Laplacian

Let Ω be an open bounded regular connected set of $\mathbb{R}^N$ $(N \geq 1)$ and $\lambda \in \mathbb{R}$. We consider here the following eigenvalue problem (EP) with Dirichlet boundary conditions,

$$(\mathrm{EP}) : \begin{cases} -d\Delta u = \lambda u & X \in \Omega, \\ u(X) = 0 & X \in \partial\Omega \end{cases}$$

whom we are reaching some couple solution (λ, u), "(eigenvalue/eigenfunction)".

Theorem 45 (Diagonalization of the Laplacian (with Dirichlet BC))
There exists an Hilbert basis $(\varphi_n)_{n\in\mathbb{N}}$ of $L^2(\Omega)$ and some real numbers

$$0 < \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \longrightarrow \infty$$

such that for all $n \geq 1$,

- $\varphi_n \in H_0^1(\Omega) \cap \mathcal{C}^\infty\left(\overline{\Omega}\right)$, and
- (λ_n, φ_n) is a couple solution for (EP).

λ_n is called an *eigenvalue* for $-d\Delta$ with Dirichlet boundary condition in Ω, and φ_n its *eigenfunction* associated.

Definition 46 (Principal eigenvalue)
The positive real λ_0 is named the *principal eigenvalue*.

Theorem 47 (Properties of the principal eigenvalue/eigenfunction)
The principal eigenvalue/eigenfunction couple (λ_0, φ_0) owns the following properties:

- λ_0 is given by the Rayleigh's formula
$$\lambda_0 = \min_{u \in H_0^1(\Omega)\setminus\{0\}} \left\{ \frac{\int_\Omega d\,|\nabla u|^2}{\int_\Omega u^2} \right\}.$$
- λ_0 is of multiplicity 1.
- φ_0 is of constant sign on Ω, whereas it is not the case for all φ_k with $k > 1$.
- If $\Omega = \mathcal{B}_R$ is a centred ball of radius R, then
  - φ_0 is radial,
  - $|X| \mapsto \varphi_0(|X|) : \mathbb{R}_+ \to \mathbb{R}$ is a decreasing function,
  - $R \mapsto \lambda_R$ is a decreasing function and $\lim_{R\to+\infty} \lambda_R = 0$.

Remark. In the content of this report we take φ_0 so that is non-negative on Ω and $\|\varphi_0\|_{L^\infty(\Omega)} = 1$.

3. Elliptic maximum principles

We state here the weak and strong elliptic maximum principles; to get further informations, one refers to [13]. Let Ω be an open regular connected set of $\mathbb{R}^N$ $(N \geq 1)$ and consider the following elliptic differential operator:

$$\mathcal{L}u := \sum_{i,j=1}^{N} a_{ij}(X)\,\partial_{x_i x_j} u + \sum_{i=1}^{N} b_i(X)\,\partial_{x_i} u + c(X)\,u.$$

Hypothesis 6: Take the following assumptions on the coefficient in $\mathcal{L}$:

- a_{ij}, b_i and c are bounded and continuous on Ω,
- for all $i, j \in [\![1; N]\!]$, $a_{ij} = a_{ji}$.

Hypothesis 7: Suppose that $-\mathcal{L}$ is *uniformly elliptic*, that is there some real $\lambda > 0$ such that for every $X \in \Omega$ and for any vector $\xi \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$, we have

$$-\sum_{i,j=1}^{N} a_{ij}\xi_i\xi_j > \lambda\,|\xi|^2.$$

Remark. Note for example that $\Delta u \overset{\text{def}}{=} \sum_{i,j=1}^{N} \delta_{ij}\partial_{x_i x_j} u$ is uniformly elliptic so one can work with $\mathcal{L}u := -\Delta u$.

Theorem 48 (Weak elliptic maximum principle)
Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ be such that $\mathcal{L}u \leq 0$ in Ω.

• If $c \equiv 0$, then $\max_{\overline{\Omega}} u$ is actually achieved on the frontier $\partial\Omega$.

• If $c \geq 0$, then $\max_{\overline{\Omega}} u$ is smaller than $\max_{\partial\Omega} \{u, 0\}$; in particular, if $\max_{\overline{\Omega}} u \geq 0$, then it is actually achieved on the frontier $\partial\Omega$.

Theorem 49 (Strong elliptic maximum principle)
Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ be such that $\mathcal{L}u \leq 0$ in Ω.

• If $c \equiv 0$ and u attains an interior maximum (*i.e.* in $\Omega = \mathring{\Omega}$), then u is constant in $\overline{\Omega}$.

• If $c \geq 0$ and u attains a non-negative interior maximum (*i.e.* in $\Omega = \mathring{\Omega}$), then u is constant in $\overline{\Omega}$.

Lemma 50 (Elliptic Hopf's lemma)
Let $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}\left(\overline{\Omega}\right)$ be such that $\mathcal{L}u \leq 0$ in Ω.

• If $c \equiv 0$ and u attains a maximum in $x_0 \in \partial\Omega$, then either

$$u \equiv u(x_0) \text{ in } \overline{\Omega} \qquad \text{or} \qquad \frac{\partial u}{\partial \nu}(x_0) > 0.$$

• If $c \geq 0$ the latter also holds under the extra assumption $u(x_0) \geq 0$.

4. Parabolic maximum principles

We state here the weak and strong parabolic maximum principles; to get further informations, one refers to [13]. Let Ω be an open regular connected set of $\mathbb{R}^N$ $(N \geq 1)$ and consider the following elliptic differential operator:

$$\mathcal{P}u := \partial_t u + \underbrace{\sum_{i,j=1}^{N} a_{ij}(X)\,\partial_{x_i x_j} u + \sum_{i=1}^{N} b_i(X)\,\partial_{x_i} u + c(X)\,u}_{:=\mathcal{L}u},$$

where the differential operator $\mathcal{L}$ satisfies hypothesis 6 and 7 defined for the elliptic maximum principles on the previous page.

Remark. Note for example one can deal with $\mathcal{P}u := \partial_t u - \Delta u$.

For $T > 0$, we define the parabolic domain and its parabolic frontier:

$$\Omega_T := (0;T] \times \Omega \qquad \text{and} \qquad \partial_p \Omega_T := \Big(\{0\} \times \overline{\Omega}\Big) \cup \Big((0;T] \times \partial\Omega\Big).$$

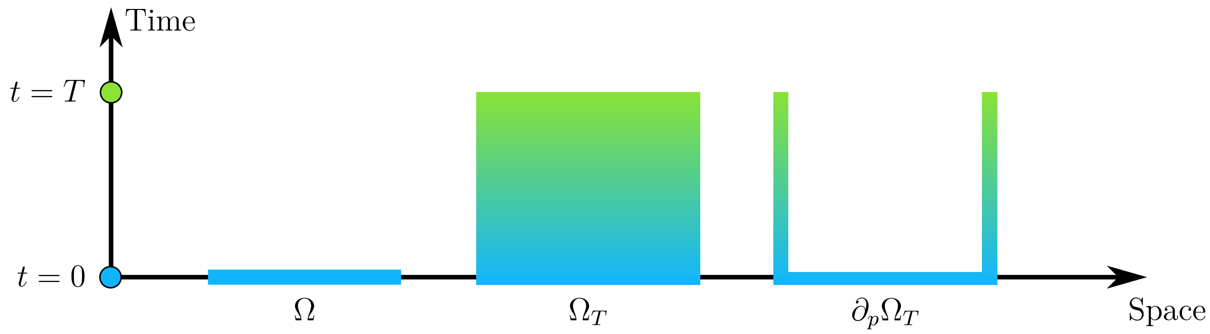


Figure F42 – *Illustration 1D of the domain Ω, the parabolic domain Ω_T and the parabolic frontier $\partial_p\Omega_T$.*

Theorem 51 (Weak parabolic maximum principle)
Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}\left(\overline{\Omega_T}\right)$ be such that $\mathcal{P}u \leq 0$ in Ω_T.

- If $c \equiv 0$, then $\max\limits_{\overline{\Omega_T}} u$ is actually achieved on the parabolic frontier $\partial_p\Omega_T$.
- If $c \geq 0$, then $\max\limits_{\overline{\Omega_T}} u$ is smaller than $\max\limits_{\partial_p\Omega_T} \{u, 0\}$; in particular, if $\max\limits_{\overline{\Omega_T}} u \geq 0$, then it is actually achieved on the parabolic frontier $\partial_p\Omega_T$.

Theorem 52 (Strong parabolic maximum principle)
Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}\left(\overline{\Omega_T}\right)$ be such that $\mathcal{P}u \leq 0$ in Ω_T.

- If $c \equiv 0$ and u attains an interior maximum at time T (*i.e.* in $\{T\} \times \Omega = \{T\} \times \overset{\circ}{\Omega}$), then u is constant in $\overline{\Omega_T}$.
- If $c \geq 0$ and u attains a non-negative interior maximum at time T (*i.e.* in $\{T\} \times \Omega = \{T\} \times \overset{\circ}{\Omega}$), then u is constant in $\overline{\Omega_T}$.

Lemma 53 (Parabolic Hopf's lemma)
Let $u \in \mathcal{C}^{1,2}(\Omega_T) \cap \mathcal{C}\left(\overline{\Omega_T}\right)$ be such that $\mathcal{P}u \leq 0$ in Ω_T.

• If $c \equiv 0$ and u attains a maximum on the frontier at time T, that is at $(T, x_0) \in \{T\} \times \partial\Omega$, then either

$$u \equiv u(T, x_0) \text{ in } \overline{\Omega_T} \qquad \text{or} \qquad \frac{\partial u}{\partial \nu}(T, x_0) > 0.$$

• If $c \geq 0$ the latter also holds under the extra assumption $u(T, x_0) \geq 0$.

Bibliography

[1] Allee, W. C. *Animal Aggregations*, vol. 2. Williams and Wilkins, 1927.

[2] Aronson, D. G., and Weinberger, H. F. Multidimensional nonlinear diffusion arising in population genetics. *Advances in Mathematics 30*, 1 (1978), 33–76.

[3] Barroux, R. Redouté, le moustique" tigre" est arrivé aux portes de paris. *Le Monde 17* (2012).

[4] Berestycki, H., Roquejoffre, J.-M., and Rossi, L. The influence of a line with fast diffusion on Fisher-KPP propagation. *J. Math. Biol. 66*, 4-5 (2013), 743–766.

[5] Cantrell, R. S., and Cosner, C. *Spatial Ecology via Reaction-Diffusion Equations.* John Wiley & Sons, 2004.

[6] Courchamp, F., Berec, L., and Gascoigne, J. *Allee Effects in Ecology and Conservation.* Oxford University Press, 2008.

[7] Fujita, H. On the blowing up of solutions fo the Cauchy problem for $u_t = \Delta u + u^\alpha$. *J. Fac. Sci. Univ. Tokyo 13* (1966), 109–124.

[8] Gatto, M., Bertuzzo, E., Mari, L., Miccoli, S., Carraro, L., Casagrandi, R., and Rinaldo, A. Spread and dynamics of the covid-19 epidemic in italy: Effects of emergency containment measures. *Proceedings of the National Academy of Sciences 117*, 19 (2020), 10484–10491.

[9] Jung, T., and Blaschke, M. Phytophthora root and collar rot of alders in bavaria: distribution, modes of spread and possible management strategies. *Plant Pathology 53*, 2 (2004), 197–208.

[10] Malthus, T. R. *An Essay on the Principle of Population*, vol. 1. Cosimo, Inc., 2013.

[11] McKenzie, H. W., Merrill, E. H., Spiteri, R. J., and Lewis, M. A. How linear features alter predator movement and the functional response. *Interface focus 2*, 2 (2012), 205–216.

[12] Okubo, A., and Levin, S. A. *Diffusion and Ecological Problems: Modern Perspectives*, vol. 14. Springer Science & Business Media, 2013.

[13] Protter, M. H., and Weinberger, H. F. *Maximum Principles in Differential Equations.* Springer Science & Business Media, 2012.

[14] Robinet, C., Imbert, C.-E., Rousselet, J., Sauvard, D., Garcia, J., Goussard, F., and Roques, A. Human-mediated long-distance jumps of the pine processionary moth in europe. *Biological invasions 14*, 8 (2012), 1557–1569.

[15] Roques, L. *Modèles de réaction-diffusion pour l'écologie spatiale.* Editions Quae, 2013.

[16] Schtickzelle, M. Pierre-François Verhulst (1804-1849). La première découverte de la fonction logistique. *Population (french edition)* (1981), 541–556.

[17] Siegfried, A. *Itinéraires de contagions: épidémies et idéologies.* FeniXX, 1960.

[18] Smith, H. Reaction-Diffusion Equations on $\mathbb{R}^N$. Online available at: `https://math.la.asu.edu/~halsmith/paraboliceqn.pdf`.

[19] Strauss, W. A. *Partial differential equations: An introduction.* John Wiley & Sons, 2007.

Notations

Functional sets

$\mathcal{C}^k(U,\mathbb{R})$ — For U an open set of $\mathbb{R}^N$ and $k \in \mathbb{N}$, set of functions $f : U \to \mathbb{R}$ such that the partial derivative

$$\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}$$

exists and is continuous on U, for all $\alpha \in \mathbb{N}^N \cap \{|\alpha| \leq k\}$. "$\mathbb{R}$" might be forgotten if ever it would be obvious.

$\mathcal{C}^\infty(U,\mathbb{R})$ — Intersection of all $\mathcal{C}^k(U,\mathbb{R})$ *i.e.*

$$\mathcal{C}^\infty(U,\mathbb{R}) := \bigcap_{k=0}^{\infty} \mathcal{C}^k(U,\mathbb{R}).$$

$\mathcal{C}_c^k(U,\mathbb{R})$ — For $k \in \mathbb{N} \cup \{\infty\}$, set of functions in $\mathcal{C}^k(U,\mathbb{R})$ which are moreover compactly supported.

$\mathcal{C}^{k,\ell}(U \times V,\mathbb{R})$ — For $k, \ell \in \mathbb{N} \cup \{\infty\}$, set of functions $U \times V \to \mathbb{R}$ which are both class $\mathcal{C}^k$ with respect to the variable in U and class $\mathcal{C}^\ell$ with respect to the variable in V.

$\mathcal{S}(\mathbb{R})$ — Schwartz space: subset of $\mathcal{C}^\infty(\mathbb{R})$ whose derivatives are rapidly decreasing, *i.e.*

$$\mathcal{S}(\mathbb{R}) := \Big\{ f\,(\mathcal{C}^\infty)(\mathbb{R},\mathbb{R}) \text{ such that } \forall n \in \mathbb{N}, \forall k \in \mathbb{N}, \lim_{|x|\to\infty} |x|^k \left|f^{(n)}(x)\right| = 0 \Big\}.$$

Gaussian or smooth compactly supported functions are for example in $\mathcal{S}(\mathbb{R})$.

$L^p(U)$	For $U \subset \mathbb{R}^N$ and $1 \leq p < \infty$, Lebesgue space of measurable functions $U \to \mathbb{R}$ whom the integral of the p^{th} power on U is finite.
$L^\infty(U)$	For $U \subset \mathbb{R}^N$, space of measurable bounded functions $U \to \mathbb{R}$.
$H^1(U)$	For $U \subset \mathbb{R}^N$, denotes the Sobolev space of $L^2(U)$ functions f which own a weak derivative f' also in $L^2(U)$.
$H_0^1(U)$	For $U \subset \mathbb{R}^N$, denotes the kernel of the trace operator: $\gamma : H^1(U) \to L^2(\partial U)$.
$\mathrm{BUC}(U)$	For $U \subset \mathbb{R}^N$, space of bounded functions $U \to \mathbb{R}$ uniformly continuous on U.
$\mathcal{D}(f)$	Domain of the function f.
$\mathrm{supp}(u)$	Support of the function u, *i.e.* the closure of the set of points where u is non-zero.
$\mathrm{Im}(f)$	Image of the function f.

Abbreviations

ODE	Ordinary Differential Equation.
PDE	Partial Differential Equation.
Fisher-KPP	Fisher, Kolmogorov, Petrovsky, Piskunov.
R-D	Reaction-Diffusion.
HTE	Hair Trigger Effect.
BC	Boundary conditions.
hyp	Hypothesis.

IBP — Integration by part.

CP — Comparison principle.

Differential operators

$\partial_t u$ — Partial derivative of u with respect to the time.

Δu — Laplacian of u.

$\dfrac{\partial u}{\partial n} = \partial_n u$ — Partial derivative of u following the normal unit exterior vector n, that is,

$$\frac{\partial u}{\partial n} = \nabla u \cdot n.$$

Norms

$|X|$ — Euclidean norm of $X \in \mathbb{R}^N$ ($N \in \mathbb{N}^\star$).

$\|f\|_{L^p(U)}$ — Usual norm on the Lebesgue space $L^p(U)$ defined by

$$\|f\|_{L^p(U)} := \left(\int_U f^p\right)^{1/p}.$$

"(U)" might be forgotten if ever it would be obvious.

$\|f\|_{L^\infty(U)}$ — Usual norm on the space $L^\infty(U)$ defined by

$$\|f\|_{L^\infty(U)} := \sup_U |f|.$$

"(U)" might be forgotten if ever it would be obvious.

Functions

δ_X — The Dirac delta in the point $X \in \mathbb{R}^N$.

OTHERS

$f * g$	Convolution product of f and g defined by $$(f * g)(x) = \int f(x-y)\, g(y)\, \mathrm{d}y.$$
$:=$ *vs* $\stackrel{\text{def}}{=}$	The symbol $:=$ denotes the first definition of an element whereas $\stackrel{\text{def}}{=}$ recalls the definition of an element defined above.
$\lvert A \rvert$	If A is a Lebesgue-measurable set, denotes then the measure of A.
$\mathrm{Lip}(f)$	If f is a Lipschitz function, denotes then the Lipschitz constant of f defined by $$\mathrm{Lip}(f) = \inf\{\ell > 0 \;/\; \forall x, y \in \mathcal{D}(f),\, \lvert f(x) - f(y)\rvert \leq \ell \lvert x - y\rvert\}.$$
$\underset{U}{\mathrm{Lip}}(f)$	Denotes then the Lipschitz constant of $f_{\lvert U}$.
$\mathcal{B}_R$	Open centred ball of radius R in $\mathbb{R}^N$.
$\mathcal{B}(X, R)$	Open ball centred on X and of radius R in $\mathbb{R}^N$.
$\mathbb{1}_A$	Indicator function of the set A.
∂A	Frontier of the set A.
$[\![1; N]\!]$	Denotes the set $\{1, \cdots N\}$.
δ_{ij}	Kronecker delta defined by $\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$
$\lesssim$	$A \lesssim B$ means that there is a positive constant $c = c(N, d, \alpha)$ such that $A \leq cB$.

Index

A
Allee . . . 9
Allee effect . . . 9
approximate identity . . . 15

B
Black death plague . . . 38
boundary conditions
  Dirichlet . . . 82
  Fourier . . . *see* Robin
  Neumann . . . 82
  Robin . . . 82

C
carrying capacity . . . 8
Cauchy problem
  global solution . . . 4
  solution . . . 4
C_{KPP} . . . 58
comparison ($\mathbb{R}^N$) . . . 22
comparison (Field-Road) . . . 46
comparison principle ($\mathbb{R}^N$) . . . 22
comparison principle (Field-Road) . . . 46
corollary
  about reaction equilibriums . . . 23
  if u_0 is sub-solution of its own PDE . . . 23
  uniqueness of the solution . . . 22

D
diffusion equation . . . 11
Duhamel's principle . . . 19

E
eigenfunction . . . 118
eigenvalue . . . 118
elliptic . . . 119
elliptic maximum principles . . . 119
equilibrium point (ODE) . . . 5
  asymptotically stable . . . 5
  not asymptotically stable equilibriums . . . 5

F
Field . . . 40
Field-Road space . . . 40
Fisher-KPP equations type . . . 24
Fourier transform . . . 117
Fujita . . . 30
Fujita's exponent . . . 30
fundamental solution . . . 83

G
growth rate *per capita* . . . 6

H
Hair Trigger Effect (HTE) . . . 25
Hair Trigger Effect on the Field-Road . . . 55
heat equation . . . *see* diffusion equation
heat kernel on $\mathbb{R}^N$ . . . 14
heat kernel on $\mathbb{R}$ . . . 14

I
intraspecific competition . . . 7
inverse Fourier transform . . . 117

K
KPP hypothesis . . . 8
KPP-type . . . 24

M
Malthus . . . 7
maximum principles
  elliptic . . . 119
  parabolic . . . 120

P
parabolic differential operator . . . 21
parabolic maximum principles . . . 120
population mass . . . 16
principal eigenvalue . . . 118

R
reaction . . . 3
  bistable model . . . 10
  linear model . . . 7
  logistic model . . . 7
  monostable degenerate model . . . 9
reaction function . . . *see* reaction
reaction-diffusion equation . . . 18
regularizing effect (heat equation) . . . 14
Road . . . 40

S
seismic lines 39
sub-solution ($\mathbb{R}^N$) 21
sub-solution (Field-Road) 46
super-solution ($\mathbb{R}^N$) 21
super-solution (Field-Road) 46

T
theorem
Aronson-Weinberger (HTE *vs.* extinction) 36
Aronson-Weinberger (Spreading speed) . . . 58
Banach-Picard fixed point 19
Berestycki *et al.* (Asymptotic spreading speed (Field-Road)) 65
Berestycki *et al.* (Spreading speed for large Road diffusion) 74
Cauchy-Lipschitz 4
comparison principle ($\mathbb{R}^N$) 22
comparison principle (Field-Road) 46
diagonalization of the Laplacian 118
Fujita (1966) 30
Global well-posedness ($\mathbb{R}^N$) 20
Global well-posedness (Field-Road) 45
Hair Trigger Effect 25
Hair Trigger Effect on the Field-Road 55
linearization (ODE) 5
Local well-posedness ($\mathbb{R}^N$) 18

U
uniformly elliptic 119

V
Verhulst 7